

THERMODYNAMIC FORMALISM MEETS ERGODIC OPTIMIZATION

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This note presents a concrete example illustrating the following connection between ergodic optimization and the thermodynamic formalism of MANEVILLE–POMEAU maps and HÖLDER continuous potentials:

(*) *If a potential undergoes a phase transition in temperature, then the DIRAC measure at the neutral fixed point uniquely maximizes the potential.*

This result is a direct consequence of [CIR25, Key Lemma and Proposition 2.10] and the concrete example appears as [CIR25, Example 1.2]. For background on ergodic optimization, see, for example, [Jen19].

The example relies on the following notation and terminology. Fix α in $(0, +\infty)$ and the MANEVILLE–POMEAU or *intermittent* map $f: [0, 1] \rightarrow [0, 1]$ defined by

$$f(x) := \begin{cases} x(1 + x^\alpha) & \text{if } x(1 + x^\alpha) \leq 1; \\ x(1 + x^\alpha) - 1 & \text{otherwise.} \end{cases}$$

Denote by \mathcal{M} the space of BOREL probability measures on $[0, 1]$ that are invariant by f and, for each ν in \mathcal{M} , by h_ν the measure-theoretic entropy of ν . For each continuous function $\varphi: [0, 1] \rightarrow \mathbb{R}$, the *pressure* $P(\varphi)$ of f for the potential φ is defined by

$$P(\varphi) := \sup \left\{ h_\nu + \int \varphi \, d\nu : \nu \in \mathcal{M} \right\}.$$

Proposition 1. *Let γ in $(0, +\infty)$ be given, and define the potential $\hat{\varphi}$ by*

$$\hat{\varphi}(x) := -x^\gamma(1 - x).$$

Then, $t \mapsto P(t\hat{\varphi})$ is real-analytic on $(0, +\infty)$.

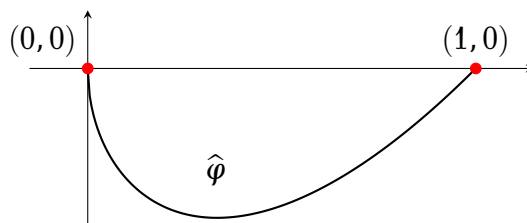


FIGURE 1. The graph of the potential $\hat{\varphi}$ with $\gamma = \frac{1}{2}$.

The potential $\hat{\varphi}$ is nonnegative and vanishes precisely at the neutral fixed point 0 and at the repelling fixed point 1. Thus, the measure δ_0 maximizes $\hat{\varphi}$, but not uniquely. The conclusion of **Proposition 1** is that the potential $\hat{\varphi}$ does not undergo a phase transition in temperature, as ensured by (*).

The remainder of this note proves **Proposition 1**. The proof relies on a couple of lemmas. Denote by x_1 the unique discontinuity point of f .

Lemma 2. For every β in $(0, +\infty)$,

$$\sup_{[0,1]} \beta \hat{\varphi} < P(\beta \hat{\varphi}).$$

In particular, the potential $\beta \hat{\varphi}$ is hyperbolic for f in the sense of [IRRL12].

Proof. Denote by x_2 the unique preimage of x_1 by f in $(0, x_1)$, and by X the maximal invariant set of f in $[x_2, 1]$. Then, $f|_X$ is uniformly expanding and topologically conjugate to the golden mean shift, via a HÖLDER continuous conjugacy. By the thermodynamic formalism for HÖLDER continuous potentials, it follows that $f|_X$ has a unique GIBBS state for the potential $\beta \hat{\varphi}|_X$ and that the measure-theoretic entropy of this measure is strictly positive. See, for example, [Bow08, Theorem 1.25]. It follows that δ_1 is not an equilibrium state of $f|_X$ for the potential $\beta \hat{\varphi}|_X$, and thus

$$P(\beta \hat{\varphi}) \geq P_{f|_X}(\beta \hat{\varphi}|_X) > h_{\delta_1}(f) + \int \beta \hat{\varphi}|_X d\delta_1 = 0.$$

Since $\hat{\varphi} \leq 0$ on $[0, 1]$, the inequality (2) follows. \square

Lemma 3. Let β be in $(0, +\infty)$ and put $g := \exp(P(\beta \hat{\varphi}) - \beta \hat{\varphi})$. Then, there is a $(1/g)$ -conformal probability measure m for f that is atom-free and satisfies $\text{supp}(m) = [0, 1]$.

Proof. Put $I := [0, 1]$ and let $\tilde{f}: \tilde{I} \rightarrow \tilde{I}$ be the continuous extension of f obtained by “doubling” the points of $\bigcup_{n=0}^{\infty} f^{-n}(x_1)$, as in [CIR25, Proof of Lemma 2.6]. Put $Y^+ := \bigcup_{n=0}^{+\infty} \tilde{f}^{-n}(x_1^+)$, and denote by $\pi: \tilde{I} \rightarrow I$ the corresponding projection and $\tilde{\varphi}: \tilde{I} \rightarrow \mathbb{R}$ the continuous extension of $\hat{\varphi}|_{\tilde{I} \setminus Y^+}$. The map \tilde{f} is a local monotone homeomorphism that is topologically exact. Furthermore, $\pi \circ \tilde{f} = f \circ \pi$, $\tilde{\varphi} = \hat{\varphi} \circ \pi$, and $\pi: \tilde{I} \setminus Y^+ \rightarrow I$ is a BOREL measurable bijection conjugating $\tilde{f}|_{\tilde{I} \setminus Y^+}$ to f .

Since \tilde{f} is topologically conjugate to the full shift in two symbols [CIR25, Lemma 2.6], the dual $\mathcal{L}_{\tilde{\varphi}}^*$ of the transfer operator $\mathcal{L}_{\tilde{\varphi}}$ of \tilde{f} for the potential $\tilde{\varphi}$ is thus continuous and has an eigenprobability \tilde{m} by the SCHAUDER–TYCHONOFF Fixed Point Theorem. Using, for every n in \mathbb{N} , the equality $(\mathcal{L}_{\tilde{\varphi}}^*)^n \tilde{m}(\tilde{I}) = \int \mathcal{L}_{\tilde{\varphi}}^n(\mathbf{1}_{\tilde{I}}) d\tilde{m}$ and that $\frac{1}{n} \log \mathcal{L}_{\tilde{\varphi}}^n \mathbf{1}_{\tilde{I}}$ converges pointwise to $P_{\tilde{f}}(\tilde{\varphi})$ as $n \rightarrow +\infty$ [CIR25, Proposition 2.9(2)], it follows that $\mathcal{L}_{\tilde{\varphi}}^* \tilde{m} = \exp(P_{\tilde{f}}(\tilde{\varphi})) \tilde{m}$ and \tilde{m} is $\exp(\beta \tilde{\varphi} - P(\beta \tilde{\varphi}))$ -conformal for \tilde{f} . The equality $\text{supp}(\tilde{m}) = \tilde{I}$ follows from the fact that \tilde{f} is topologically exact on $(0, 1]$ [CIR25, Lemma 2.8].

To prove that \tilde{m} is atom-free, let \tilde{x} in \tilde{I} be given. For every k in \mathbb{N} ,

$$\tilde{m}(\{x\}) = \exp(-(kP(\hat{\varphi}) - S_k(\tilde{\varphi})(\tilde{x}))) \tilde{m}(\{\tilde{f}^k(x)\}) \leq \exp\left(-k \left(P(\hat{\varphi}) - \sup_{[0,1]} \hat{\varphi}\right)\right) \tilde{m}(\{\tilde{f}^k(\tilde{x})\}).$$

Letting $k \rightarrow +\infty$ and using (2) in Lemma 3 yields $\tilde{m}(\{\tilde{x}\}) = 0$. This proves that \tilde{m} is atom-free.

Put $m := \pi_* \tilde{m}$. Since $\text{supp}(\tilde{m}) = \tilde{I}$ and \tilde{m} is atom-free, $\text{supp}(m) = I$ and m is atom-free and $(1/g)$ -conformal for f . \square

Proof of Proposition 1. Let β be in $(0, +\infty)$, put $g := \exp(P(\beta \hat{\varphi}) - \beta \hat{\varphi})$ as in Lemma 3, and put $\gamma_* := \min\{1, \gamma\}$. The functions $\hat{\varphi}$ and g belong to $C^{\gamma_*}([0, 1], \mathbb{R})$ and g satisfies $\sup_{[0,1]} g < 1$

by **Lemma 2**. Let m be the atom-free $(1/g)$ -conformal probability measure for f satisfying $\text{supp}(m) = [0, 1]$ given by **Lemma 3**.

The first step is to verify the hypotheses of [LRL14, Theorem 1 in §4.2], which is a reformulation of results of KELLER [Kel85, Theorems 3.2 and 3.3]. Hypotheses H1 and H2 of [LRL14, Theorem 1 in §4.2] follow automatically from the fact that every function in $C^{\gamma_*}([0, 1], \mathbb{R})$ is of bounded $(1/\gamma_*)$ -variation and that m is $(1/g)$ -conformal for f . Hypothesis H3 of [LRL14, Theorem 1 in §4.2] can be shown as in the proof of [LRL14, Lemma 5.1]. Thus, the hypotheses of [LRL14, Theorem 1 in §4.2] are satisfied. Using that f is topologically exact on $(0, 1]$ [CIR25, Lemma 2.8], [LRL14, Corollary 4.4(1)] yields that $t \mapsto P(t\varphi)$ is real-analytic at $t = \beta$. \square

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