STATISTICAL PROPERTIES OF REAL AND COMPLEX MAPS

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Contents

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1. The probabilistic viewpoint of dynamical systems

In general terms, a *discrete time dynamical system* is a map T acting on a space X,

$$T : X \to X.$$

Usually X represents the state space of a system, and T represents the transformation law of the states after a unit of time. For each integer $n \geq 1$ we denote by

$$T^n := \underbrace{T \circ T \circ \cdots \circ T}_{n}$$

the n-th iterate of T. We also use T^0 to denote the identity on X. So the map $T^n : X \to X$ represents the evolution at time n of the states.

Given a point x_0 in X, the *orbit of x_0 for T* is the sequence of points in X,

$$x_0, T(x_0), T^2(x_0), \ldots.$$

The point x_0 is the *initial condition* of the orbit.

Roughly speaking, one of the central goals of the dynamical systems theory is to *describe the long term behavior of most orbits of a typical dynamical systems.*

In some cases the transformation T is simple, but the dynamical system that it generates can be difficult to understand. One of the emblematic examples is the complex quadratic family: Given a complex number c, consider the quadratic polynomial $P_c(z) := z^2 + c$, seen as a transformation of the complex plane $\mathbb{C}$:

$$\begin{array}{cccc} P_c : & \mathbb{C} & \to & \mathbb{C} \\ & z & \mapsto & P_c(z) := z^2 + c. \end{array}$$

Observe that for each value of the parameter c we have a different discrete time dynamical system. General aspects of the dynamics of complex polynomials can be found, for example, in [CG93, Mil06].

Let us fix a parameter c in $\mathbb{C}$. The interesting part of the dynamics of P_c occurs in the set

$$K_c := \left\{ z_0 \in \mathbb{C} : (P_c^n(z_0))_{n \geq 1} \text{ is bounded} \right\},$$

that is called the *filled Julia set of P_c*. It is a compact set, and its boundary $J_c := \partial K_c$ is called the *Julia set of P_c*.

The complexity of the dynamics of P_c is reflected in the complexity of J_c. For example, Figure 1.1 represents the Julia set of parameter $c = i$, for which we have $J_i = K_i$. It is a dendrite, and therefore its topological dimension is equal to 1. However, the Hausdorff dimension of J_i is strictly bigger than 1, so it is a fractal in the sense of Mandelbrot.

Another interesting example is the filled Julia set of parameter

$$c_0 := \frac{\lambda_0}{2} - \frac{\lambda_0^2}{4}, \text{ where } \lambda_0 := \exp\left(2\pi i \cdot \frac{1 + \sqrt{5}}{2}\right),$$

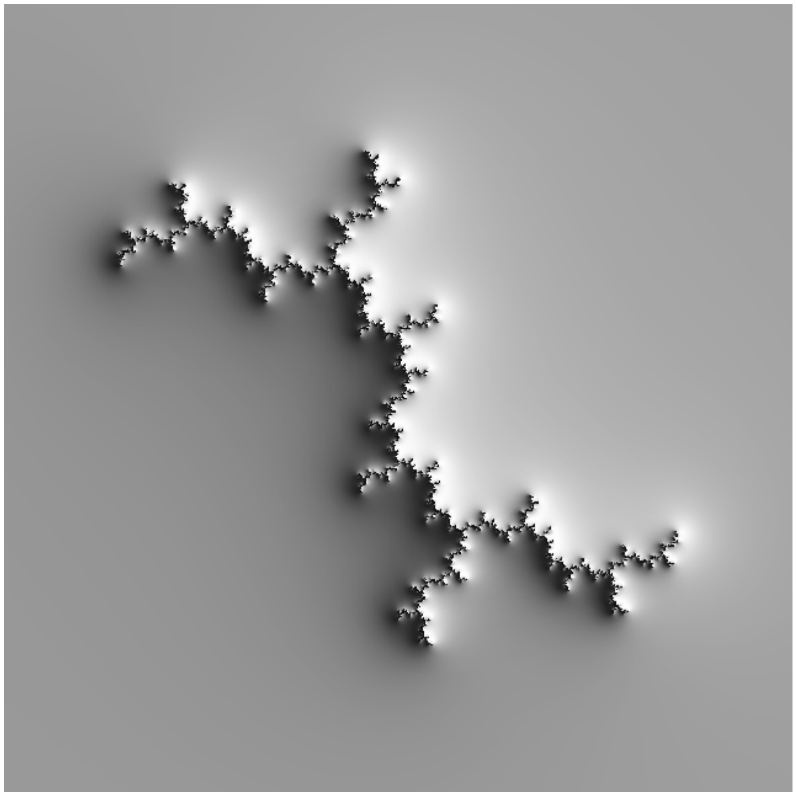

FIGURE 1.1. Dendrite Julia set, from Tomoki Kawahira's galery.

represented in Figure 1.2. One of the features of P_{c_0} is that it has a fixed point at which the derivative of P_{c_0} is equal to λ_0: The point $z_0 := \frac{\lambda_0}{2}$ satisfies

$$P_{c_0}(z_0) = z_0 \text{ and } P'_{c_0}(z_0) = \lambda_0.$$

This implies that the filled Julia set K_{c_0} of P_{c_0} contains a Siegel disk around z_0.

Figure 1.3 represents the filled Julia set of parameter

$$c_1 := \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4}, \text{ where } \lambda_1 := \exp\left(2\pi i \cdot \frac{55}{89}\right),$$

is close to c_0. The filled Julia sets K_{c_0} and K_{c_1} are very similar, but the Julia set J_{c_1} fills a larger space than J_{c_0}; it occupies a good part of the interior of K_{c_0}. Something similar occurs with other well-chosen perturbations of the parameter c_0. This phenomenon was one of the ingredients used by Buff and Chéritat to prove the existence of a parameter whose Julia set has positive Lebesgue measure, see [BC06]. In this way they solved a problem proposed in the work of Fatou, about a century ago.

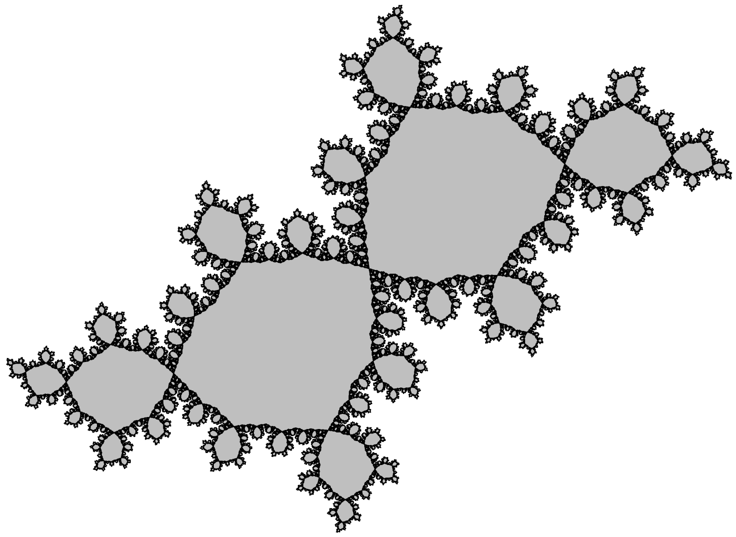

Figure 1.2. Filled Julia set with a Siegel disk, from Arnaud Chéritat's gallery.

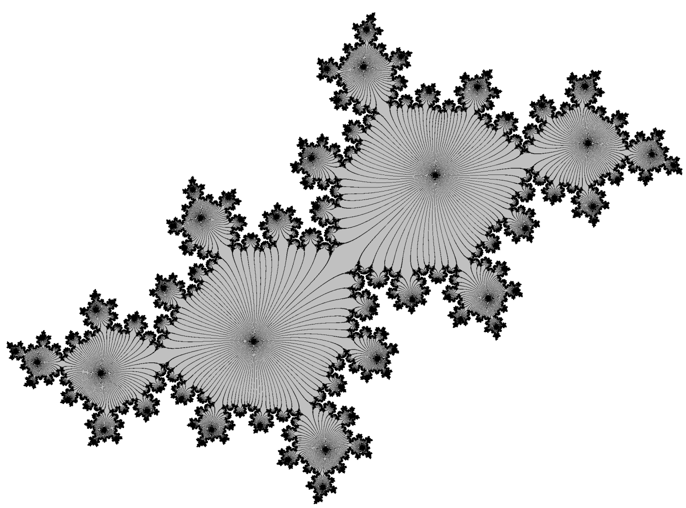

Figure 1.3. Filled Julia set with a parabolic fixed point, from Arnaud Chéritat's gallery.

Finally, Figure 1.4 represents the *Mandelbrot set*: The set of all those parameters c in $\mathbb{C}$ for which the corresponding Julia set is connected.

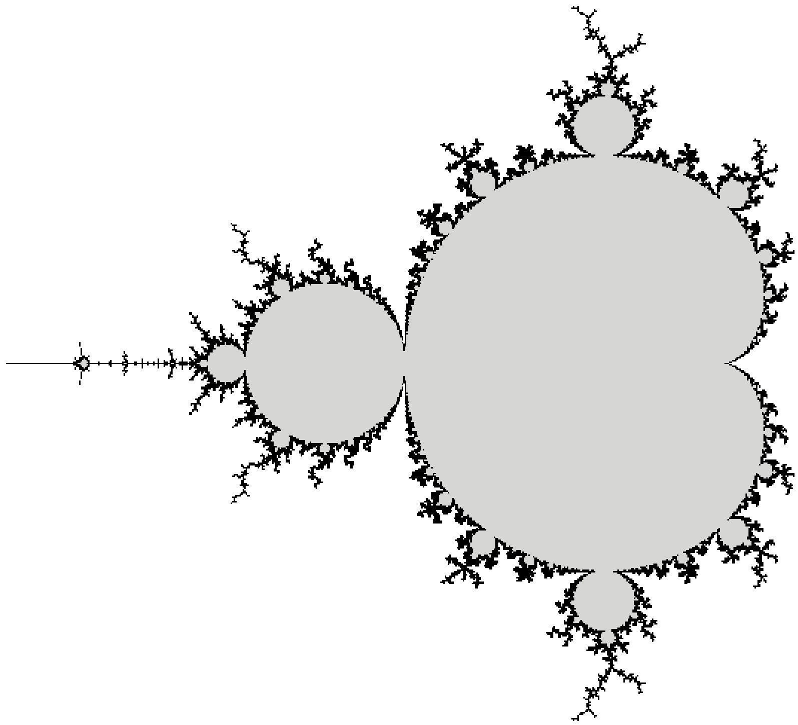

FIGURE 1.4. The Mandelbrot set, from Curtis McMullen's gallery.

1.1. **The probabilistic viewpoint.** Paradoxically, the best description we have of some discrete time dynamical systems is in probabilistic terms. This is best explained introducing the following terminology.

Definition 1.1. Let X be a topological space, and ν a Borel probability measure on X. Then we say that a sequence of points $(x_n)_{n=0}^{+\infty}$ in X is *equidistributed with respect to* ν, or that it *describes the asymptotic distribution of* $(x_n)_{n=0}^{\infty}$, if for every continuous function $\varphi : X \to \mathbb{R}$ we have

$$\lim_{n\to+\infty} \frac{\varphi(x_0) + \cdots + \varphi(x_{n-1})}{n} = \int_X \varphi \, d\nu.$$

When considering a map T acting on a topological space X, we will be interested in those measures ν that describe the asymptotic distribution of 1 or many orbits. If the orbit of a point x_0 in X is equidistributed with respect to a measure ν, then for every continuous function $\varphi : X \to \mathbb{R}$ we have

$$\lim_{n\to+\infty} \frac{\varphi(x_0) + \cdots + \varphi\left(T^{n-1}(x_0)\right)}{n} = \int_X \varphi \, d\nu. \tag{1.1}$$

The left hand side of the equation is the *time average of φ along the orbit of x_0*, and the right hand side is the *space average of φ with respect to ν*. Therefore, in this case equidistribution ensures that *the time average coincides with the space average*.

To understand (1.1) better, consider a measurable subset A of X whose boundary ∂A has measure zero with respecto to ν; that is $\nu(\partial A) = 0$. Then a straightforward approximation argument shows that in (1.1) we can replace φ by the indicator function $\mathbb{1}_A$ of A. For each integer $n \geq 1$, the number

$$\frac{\mathbb{1}_A(x_0) + \mathbb{1}_A(T(x_0)) + \cdots + \mathbb{1}_A\left(T^{n-1}(x_0)\right)}{n} = \frac{1}{n}\#\left\{j \in \{0, \ldots, n-1\} : T^j(x_0) \in A\right\}$$

is equal to the proportion of times j in $\{0, \ldots, n-1\}$ for which $T^j(x_0)$ is in A. Therefore (1.1) ensures that *the average time the orbit of x_0 visits A is equal to $\nu(A)$ in the limit.*

Figure 1.5 represents part of an orbit of an automorphism of the $K3$ surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given in affine coordinates by:

$$(1 + x^2)(1 + y^2)(1 + z^2) + 8xyz = 2.$$

automorphism. The automorphism preserves an area form, but it is not known whether there is an orbit that is equidistributed with respect to this measure, see [McM02] for further details.

1.2. **Equidistribution in the circle.** In this section we consider the circle $\mathbb{R}/\mathbb{Z}$ endowed with the additive group structure inherited from $\mathbb{R}$, and with the probability measure Leb induced by the Lebesgue measure on $\mathbb{R}$.

Given α in $\mathbb{R}/\mathbb{Z}$, the *rotation of angle α*, or the *translation by α*, is defined by

$$\begin{array}{rccc} T_\alpha : & \mathbb{R}/\mathbb{Z} & \to & \mathbb{R}/\mathbb{Z} \\ & x & \to & T_\alpha(x) := x + \alpha. \end{array}$$

Weyl's equidistribution theorem. *Let α in $\mathbb{R}/\mathbb{Z}$ be irrational, and let $T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the rotation of angle α. Then for every x_0 in $\mathbb{R}/\mathbb{Z}$ the orbit $\left(T_\alpha^j(x_0)\right)_{n=0}^{+\infty}$ of x_0 for T_α is equidistributed with respect to* Leb.

For a hint on the proof, see Exercise 1.1.

Figure 1.6 represents a part of an orbit for the rotation of angle equal to the golden mean number, $\alpha = \frac{\sqrt{5}+1}{2}$.

An important feature of irrational rotations is that *every* orbit is equidistributed with respect to the same measure. This is rather exceptional; usually there is a nonempty set of initial conditions for which equidistribution fails. For a concrete example, consider *angle doubling map*

$$\begin{array}{rccc} m_2 : & \mathbb{R}/\mathbb{Z} & \to & \mathbb{R}/\mathbb{Z} \\ & x & \mapsto & m_2(x) := 2x. \end{array}$$

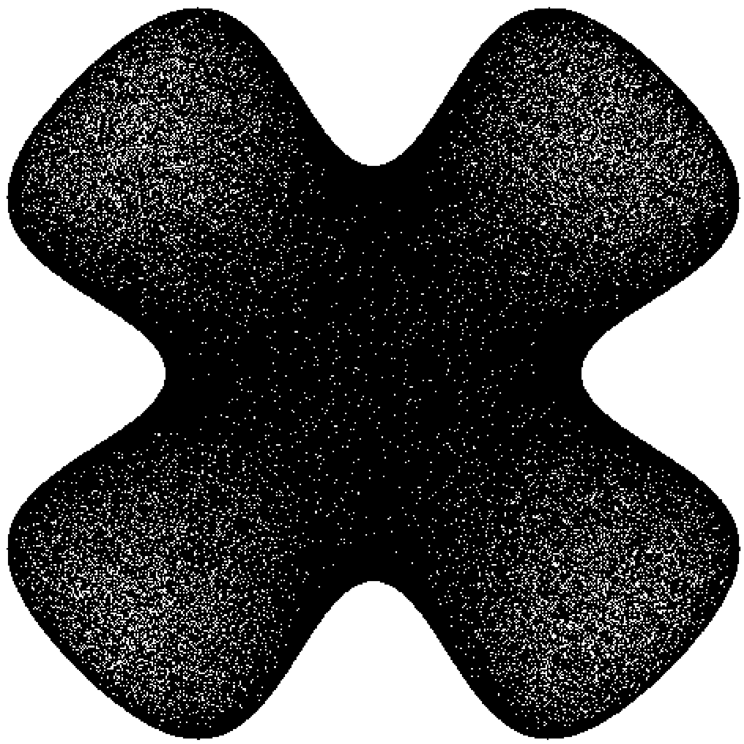

FIGURE 1.5. Orbit of a $K3$ surface automorphism, from Curtis McMullen's gallery.

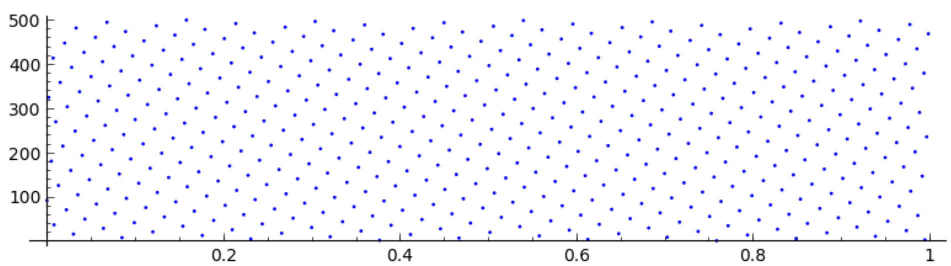

FIGURE 1.6. Orbit up to time 500 for a the rotation of angle $\alpha = \frac{\sqrt{5}+1}{2}$. The vertical coordinate is the time, and the horizontal the circle, represented by the interval $[0, 1]$.

For this map, equidistribution holds for a set of initial conditions that has full measure with respect to Leb, but it does not hold for every initial condition, see Exercise 1.2.

Theorem 1. *For every initial condition x_0 in a subset of $\mathbb{R}/\mathbb{Z}$ of full measure with respect to* Leb*, the orbit $\left(m_2^j(x_0)\right)_{n=0}^{+\infty}$ of x_0 for m_2 is equidistributed with respect to* Leb.

This result can be easily obtained form the Pointwise Ergodic Theorem using Fourier series, see Exercise 1.2.

1.3. **Logistic family.** In this section we consider the *logistic family* $(f_\lambda)_{\lambda \in (0,4]}$, defined for a parameter λ in $(0, 4]$ by

$$\begin{array}{rccc} f_\lambda : & [0,1] & \to & [0,1] \\ & x & \mapsto & f_\lambda(x) := \lambda x(1-x). \end{array}$$

The logistic family has attracted a lot of interest since in 1976 the biologist Robert May proposed it as a model for population dynamics.

Figure 1.7 represents the graph of f_λ for $\lambda = 3.74$, and Figure 1.8 a part of an orbit for this map. After some time, the orbit is attracted

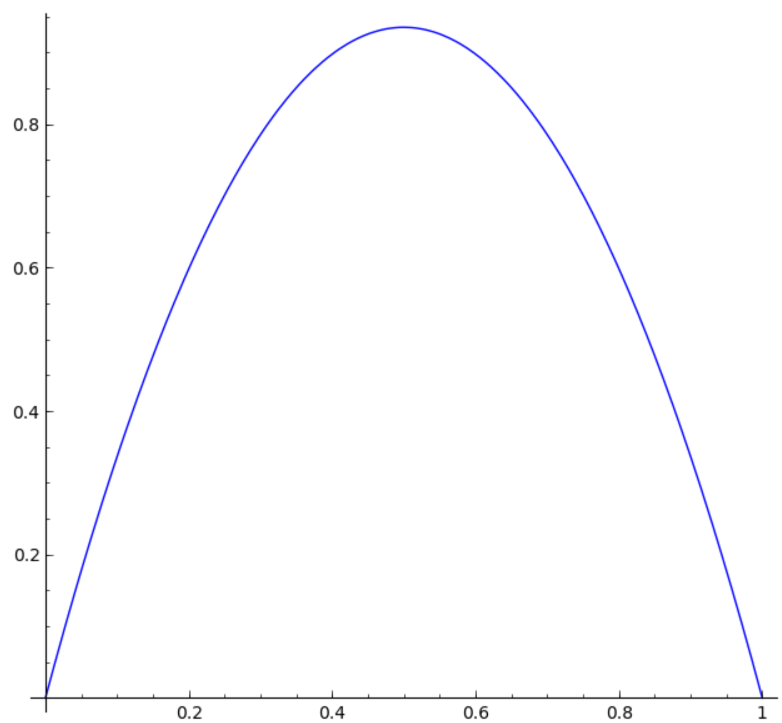

FIGURE 1.7. Graph of the logistic map f_λ, for $\lambda = 3.74$.

by an attracting periodic orbit of period 5.* It can be shown that there

*It is probably very difficult to show by hand that for $\lambda = 3.74$ the map f_λ has an attracting periodic orbit. For parameters that are closer to 4, this is hard even with a (top notch) computer. To illustrate this computational difficulty, in one of his papers

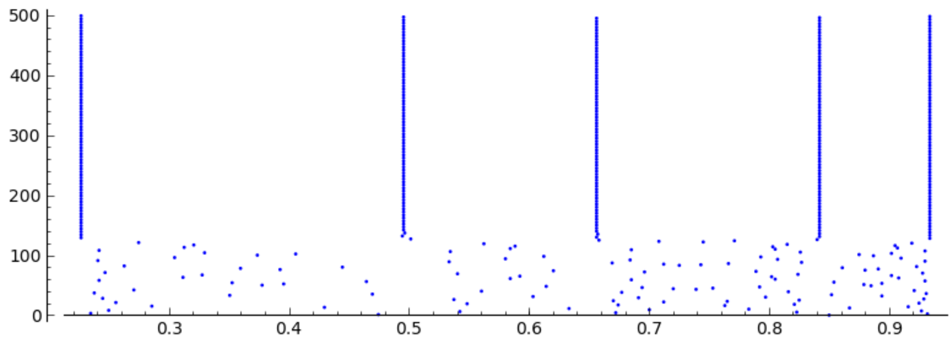

FIGURE 1.8. Orbit up to time 500 for the logistic map f_λ, with $\lambda = 3.74$. The vertical coordinate represents time, and the horizontal the interval $[0,1]$.

is a set of full Lebesgue measure in $[0,1]$ of initial conditions whose orbits are equidistributed with respect to a measure supported on the attracting periodic orbit. The parameter $\lambda = 3.74$ is an example of a *regular* parameter, see §1.5.

For $\lambda = 4$, the logistic map f_4 is known as the *Ulam and von Neumann map*, who proposed it in 1947 as a pseudo-random number generator. Figure 1.9 represents a part of an orbit for this map. It is less regular than the

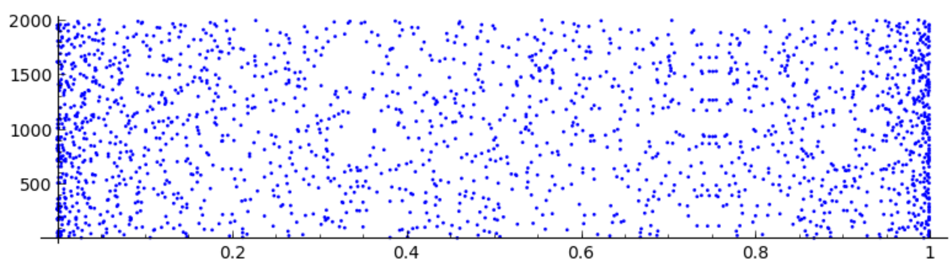

FIGURE 1.9. Orbit up to time 2000 for the Ulam and von Neumann map. The vertical coordinate represents time, and the horizontal the interval $[0,1]$.

orbit for the rotation of angle $\alpha = \frac{\sqrt{5}+1}{2}$ of Figure 1.6. However, in §1.4 we will show that there is in fact some regularity: There is a set of full Lebesgue measure in $[0,1]$ of initial conditions whose orbits are equidistributed with respect to the measure $\frac{dx}{\pi\sqrt{x(1-x)}}$. The parameter $\lambda = 4$ is an example of a *stochastic* parameter, see §1.5.

McMullen speculates that the problem of deciding whether a parameter like $\lambda = 3.99999$ has an attracting periodic orbit is probably undecidable.

Figure 1.10 represents the bifurcation diagram of the logistic family. The vertical coordinate is the parameter λ, from $\lambda \sim 3.1$ in the bottom, to $\lambda = 4$ in the top. On the horizontal line of height λ, it is represented the limit of the orbit of $x_0 = 1/2$. For example, for λ close to 0 this set is reduced to a single point, and for λ close to 4 the limit seems to be an interval, see Exercise 1.3.

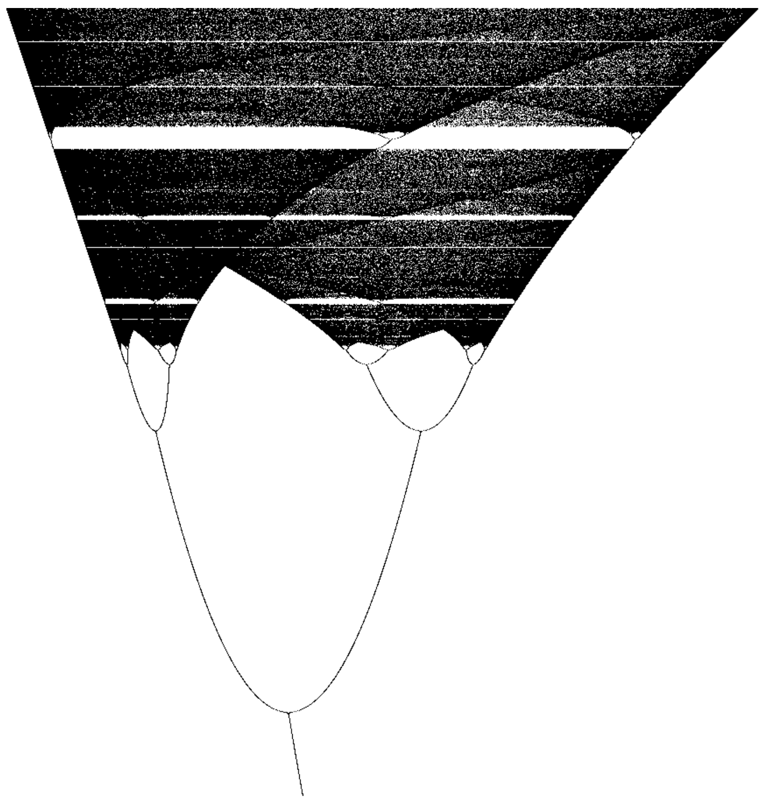

FIGURE 1.10. Bifurcation diagram of the logistic family.

1.4. **Histograms.** In this section we analyze the statistical properties of the Ulam and von Neumann map, $f_4(x) = 4x(1-x)$. We start with a graphic representation of the distribution of an orbit, usually known as "histogram".

Consider an initial condition x_0, its orbit for f_4 up to a certain time T, and divide the interval $[0,1]$ in to a number N of intervals of the same length. With this data we define for each i in $\{0,\ldots,N-1\}$ the number

$$v_i := \left\{ j \in \{0,\ldots,T-1\} : f_4^j(x_0) \in \left[\frac{i}{N}, \frac{i+1}{N}\right) \right\},$$

and consider the graph that over each interval $\left[\frac{i}{N}, \frac{i+1}{N}\right)$ has height equal to $v_i\frac{N}{T}$. We added the factor $\frac{N}{T}$ so that the area below the graph is equal to 1.

Figure 1.11 is the histogram with $x_0 = 1/\pi$, $T = 2000$, and $N = 100$. Observe that this piece of orbit spends more time near the end points of $[0, 1]$, something that can also be appreciated in Figure 1.9.

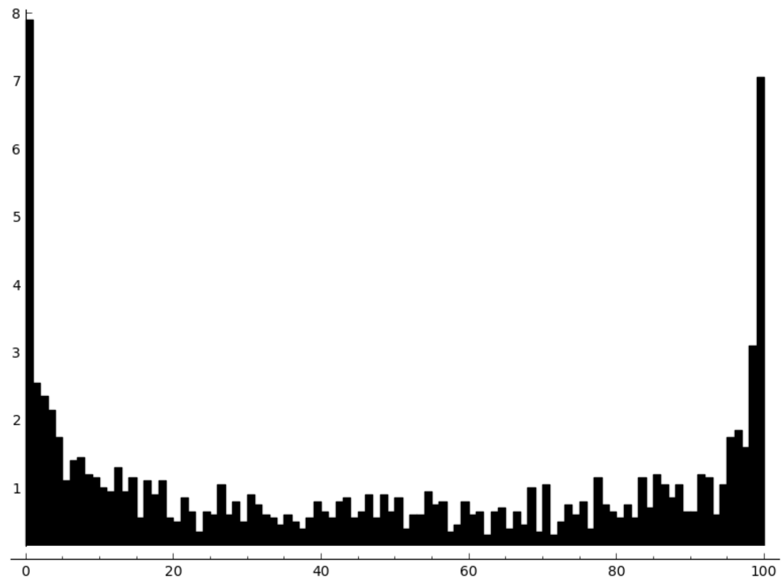

FIGURE 1.11. Histogram of an orbit up to time 2000 for the Ulam and von Neumann map.

In what follows we use the Pointwise Ergodic Theorem to show that the histograms of the orbits of f_4 converge to the graph of the function $x \mapsto \frac{1}{\pi\sqrt{x(1-x)}}$, provided that the initial condition belongs to a certain set of full Lebesgue measure in $[0, 1]$, and that we first let $T \to +\infty$ and afterwards we let $N \to +\infty$. Figure 1.12 is the graph of the histogram with $x_0 = 1/e$, $T = 20000$, and $N = 200$, together with the graph of the function $x \mapsto \frac{1}{\pi\sqrt{x(1-x)}}$.

Recall that for a measure space $(X, \mathscr{F}, \mu)$, a measurable subset Y has *zero measure* is $\mu(Y) = 0$, and it has *full measure* if $X \setminus Y$ has zero measure. Given a map $T : X \to X$, a subset Y of X is *invariant*, if $T^{-1}(Y) = Y$.

Definition 1.2. Let $(X, \mathscr{F}, \mu)$ be a measure space and $T : X \to X$ a measurable map. We say that T is *ergodic with respect to* μ, or that μ *is ergodic for* T, if for every measurable subset of X that is invariant by T has zero or total measure.

Given a measure space $(X, \mathscr{F}, \mu)$, a measurable map $T : X \to X$ *preserves measure* if for every measurable set A of X we have

$$\mu\left(T^{-1}(A)\right) = \mu(A).$$

The following is an immediate consequence of the Pointwise Ergodic Theorem.

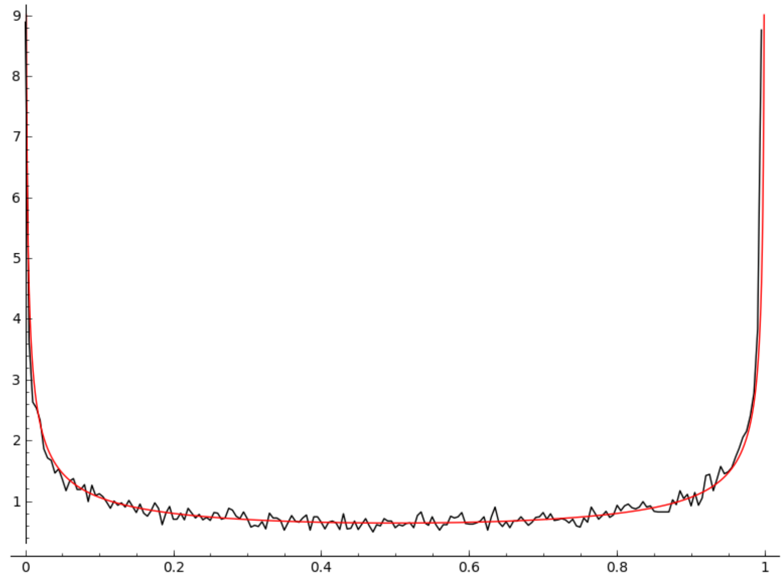

FIGURE 1.12. Graph of the function $x \mapsto \frac{1}{\pi\sqrt{x(1-x)}}$, together with the graph of the histogram of an orbit for the Ulam and von Neumann map.

Theorem 2. *Let $(X, \mathscr{F}, \mu)$ be a measure space and $T : X \to X$ a measurable map the preserves measure. Then for every measurable function $\varphi : X \to \mathbb{R}$ satisfying $\int |\varphi| d\mu < +\infty$ there is a subset of full measure of X where*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ T^j = \int \varphi d\mu.$$

To apply this result to the Ulam and von Neumann map f_4, consider the interval $[0, 1]$ endowed with its Borel σ-algebra $\mathscr{B}$, and the probability measure ν defined by

$$\nu := \frac{dx}{\pi\sqrt{x(1-x)}}. \tag{1.2}$$

A straightforward computation shows that for every interval I contained in $[0, 1]$ we have

$$\nu\left(f_4^{-1}(I)\right) = \nu(I). \tag{1.3}$$

This implies that the Ulam and von Neumann map f_4 preserves ν. Therefore, the following result allow us to apply Theorem 2 to f_4 and ν.

Lemma 1.3. *The Ulam and von Neumann map is ergodic with respect to the measure* (1.2)*, and therefore with respect to the Lebesgue measure on* $[0, 1]$.

We are now in position to prove that the histograms of the orbits of f_4 converge to the function $x \mapsto \frac{1}{\pi\sqrt{x(1-x)}}$. Let $N \geq 1$ be an integer, and for every i in $\{0, \ldots, N-1\}$ let $X_{N,i}$ be the subset of $[0,1]$ of full measure for ν given by Theorem 2 with $\varphi = \mathbb{1}_{[\frac{i}{N}, \frac{i+1}{N})}$. It follows that

$$X_N := \bigcap_{i=0}^{N-1} X_{N,i}$$

is a subset of $[0,1]$ of full measure for ν, and therefore for the Lebesgue measure on $[0,1]$.

Fix x_0 in $\bigcap_{N=1}^{\infty} X_N$ and an integer $N \geq 1$. Given an integer $T \geq 1$, for each i in $\{0, \ldots, N-1\}$ define

$$v_{i,T} := \left\{ j \in \{0, \ldots, T-1\} : f_4^j(x_0) \in \left[\frac{i}{N}, \frac{i+1}{N}\right) \right\}.$$

Then Theorem 2 implies that

$$\lim_{T \to +\infty} \frac{v_{i,T}}{T} = \int \mathbb{1}_{[\frac{i}{N}, \frac{i+1}{N})} \, d\nu = \int_{\frac{i}{N}}^{\frac{i+1}{N}} \frac{dx}{\pi\sqrt{x(1-x)}}.$$

This proves that, if we fix x_0 and N and let $T \to +\infty$, the histogram of the orbit of x_0 for f_4 up to time T converges to the graph of the function

$$\sum_{i=0}^{N-1} \left(N \int_{\frac{i}{N}}^{\frac{i+1}{N}} \frac{dx}{\pi\sqrt{x(1-x)}} \right) \mathbb{1}_{[\frac{i}{N}, \frac{i+1}{N})}.$$

To conclude, note that when $N \to +\infty$ this last function converges to $x \mapsto \frac{1}{\pi\sqrt{x(1-x)}}$.

1.5. **Physical measures and absolutely continuous invariant probabilities.** In the equidistribution results we have seen in §§1.2, 1.3, and 1.4, there is a unique measure that describes the asymptotic distribution of almost every orbit. This motivates the concept of "physical measure", that we introduce now.

In many cases of interest, the space X is endowed with a natural reference measure. For example, the interval $[0,1]$ is endowed with the restriction of the Lebesgue measure on $\mathbb{R}$, and the circle $\mathbb{R}/\mathbb{Z}$ is endowed with the measure Leb induced by the Lebesgue measure on $\mathbb{R}$. More generally, for a differentiable manifold we can consider a volume form as reference measure.

Definition 1.4. Let X be a topological space and $T : X \to X$ a continuous map. The *basin* of a Borel probability measure ν on X is the set of all initial conditions in X whose orbit for T is equidistributed with respect to ν. We say that ν is a *physical measure with respect to a Borel measure μ on X*, if the basin of ν has positive measure with respect to μ.

With this terminology, Weyl's Equidistribution Theorem stated in §1.2, can be formulated as follows: For an irrational rotation, the basin of Leb is equal to the whole of $\mathbb{R}/\mathbb{Z}$. In particular, the measure Leb is a physical measure with respect to itself. Similarly, Theorem 1 can be formulated as follows: For the angle doubling map m_2, the measure Leb is a physical measure with respect to itself.

Combining Theorem 2 with Lemma 1.3 we obtain that for the Ulam and von Neumann map $f_4(x) = 4x(1-x)$, the measure $\frac{dx}{\pi\sqrt{x(1-x)}}$ is a physical measure with respect to the Lebesgue measure on $[0,1]$.

Consider a parameter λ in $(0,4]$ such that the logistic map f_λ has an attracting periodic orbit, like the parameter $\lambda = 3.74$ considered in Figure 1.8. More precisely, suppose that there is a point x_0 in $[0,1]$ and an integer $n \geq 1$, such that

$$f_\lambda^n(x_0) = x_0 \text{ y } |Df_\lambda^n(x_0)| \leq 1.$$

Then it can be shown that the basin of

$$\frac{1}{n}\left(\delta_{x_0} + \cdots + \delta_{f_\lambda^{n-1}(x_0)}\right)$$

has full Lebesgue measure on $[0,1]$. In particular, this measure is a physical measure with respect to the Lebesgue measure. We say that a parameter with this property is *regular*.

The following result gives a criterion to find physical measures. It is an immediate consequence of Theorem 2.

Proposition 1.5. *Let X be a topological space endowed with a reference measure ν, and let $T : X \to X$ be continuous and ergodic with respect to μ. If ν is a Borel probability measure on X that is invariant for T and that is absolutely continuous with respecto to μ, then the basin of ν has full measure with respect to ν, and therefore ν is a physical measure with respect to μ.*

Figure 1.13 represents the density of a measure on the Riemann sphere that is invariant and ergodic with respect to a certain complex rational map. By Proposition 1.5 it is a physical measure.

The following result will allow us to apply the criterion of Proposition 1.5 to all logistic maps. It is a vast generalization of Lemma 1.3.

Theorem 3 (Blokh-Lyubich, [BL91])**.** *Every logistic map is ergodic with respect to the Lebesgue measure.*

Combined with the fact that the basin of a measure is an invariant set, this result implies the following as an immediate consequence.

Corollary 1.6. *For a logistic map, the basin of a physical measure has full Lebesgue measure in $[0,1]$. In particular, a logistic map has at most 1 physical measure.*

By Proposition 1.5 we also have as an immediate property:

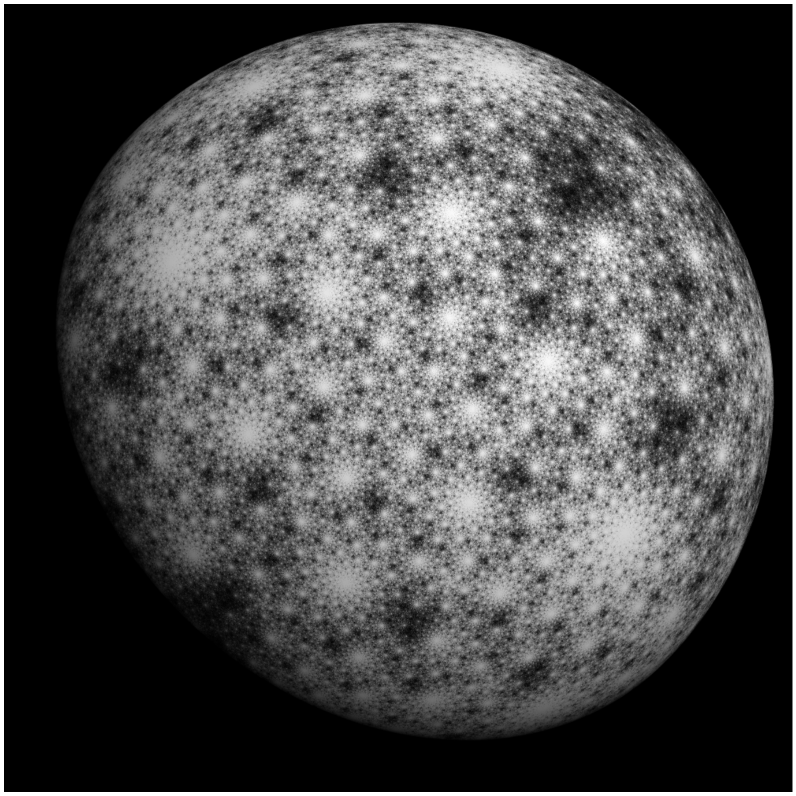

FIGURE 1.13. Density of a measure on the Riemann sphere that is a physical measure for a certain complex rational map, from Arnaud Chéritat's gallery.

Corollary 1.7. *For a logistic map, every invariant Borel probability measure that is absolutely continuous with respect to the Lebesgue measure is a physical measure.*

A parameter λ in $(0, 4]$ is *stochastic*, if f_λ admits an invariant Borel probability measure that is absolutely continuous with respect to the Lebesgue measure. By Corollaries 1.6 and 1.7, such a measure is physical with respect to the Lebesgue measure, and its basin has full measure in $[0, 1]$. On the other hand, the considerations in §1.4 for the Ulam and von Neumann map apply without change to every stochastic parameter. These considerations imply that for each stochastic parameter λ, the histograms of almost every orbit for f_λ converge to the density of the physical measure.

The parameter $\lambda = 4$ is stochastic, since the measure (1.2) is absolutely continuous with respect to the Lebesgue measure, and it is invariant by f_4, see §1.4.

Observe that for a regular parameter, the corresponding physical measure is not absolutely continuous with respect to the Lebesgue measure, since it is supported on a finite set.

We end this introduction with the following result of Lyubich, that shows that almost every logistic map has a physical measure. See also [Lyu00] for an overview.

Theorem 4 (Lyubich, [Lyu02])**.** *With respect to the Lebesgue on* $(0, 4]$*, almost every parameter is regular os stochastic.*

1.6. **Exercises.**

Exercise 1.1. Using that trigonometric polynomials are dense in the space of continuous functions, prove Wyel's equidistribution theorem.

Exercise 1.2. Describe the action of the angle doubling map m_2 on Fourier series, and conclude that m_2 is ergodic with respect to Leb. Deduce from this Theorem 1 using the Pointwise Ergodic Theorem. Moreover, show that if U is a non-empty open set of $\mathbb{R}/\mathbb{Z}$, then the orbit of each point in

$$K(U) := \{x \in \mathbb{R}/\mathbb{Z} : \text{ for every integer } n \geq 0, m_2^n(x) \notin U\}$$

is *not* equidistributed with respect to Leb. Conclude that the set of those initial conditions whose orbit is not equidistributed with respect to Leb has full Hausdorff dimension.

Exercise 1.3 (Basics of the logistic family). The purpose of this exercise is to describe all those parameters λ in $(0, 4]$ for which the logistic map $f_\lambda(x) = \lambda x(1 - x)$ has an attracting periodic point of period 1 or 2, and for each such parameter describe the dynamics of f_λ.

A. Prove that for λ in $[0, 1]$ the orbit of every point in $[0, 1]$ for the logistic map f_λ converges to the fixed point $x = 0$.
B. Prove that for λ in $(1, 3]$ the orbit of every point in the open interval $(0, 1)$ for the logistic map f_λ converges to the fixed point $x = 1 - \lambda^{-1}$.
C. Set $\lambda_0 := 1+\sqrt{5} \in (3, 4)$ and prove that for λ in $(0, 4]$ we have $f_\lambda^2(1/2) \leq 1/2$ if and only if λ is in $(0, \lambda_0]$, and that equality holds only when $\lambda = \lambda_0$.
D. Prove that for λ in $(3, 4]$ the map f_λ has a unique periodic orbit of period 2. Moreover, prove that for λ in $(3, \lambda_0]$, the orbit of every point in $(0, 1)$ that is not eventually mapped to the fixed point $1-\lambda^{-1}$ is attracted to the periodic orbit of period 2.

Exercise 1.4. The following exercise is devised to analyze the dynamics of a regular parameter having an attracting periodic point of period 3.

A. Show that there is a parameter λ_1 in $(\lambda_0, 4]$ such that $f_{\lambda_1}^3(1/2) = 1/2$, and prove that

$$f_{\lambda_1}^2(1/2) < 1/2 < f_{\lambda_1}(1/2).$$

B. Note that for this parameter we have $Df_{\lambda_1}^3(1/2) = 0$, so the orbit $\mathcal{O}$ of $x = 1/2$ for f_{λ_1} is attracting, and that the intervals

$$A_0 := \left(f_{\lambda_1}^2(1/2), 1/2\right) \text{ and } B_0 := (1/2, f_{\lambda_1}(1/2))$$

satisfy $f_{\lambda_1}(A_0) = B_0$ and $f_{\lambda_1}(B_0) = A_0 \cup B_0 \cup \{1/2\}$.

C. Show that there are closed intervals A and B such that

$$A \subset A_0, B \subset B_0, B \subset f_{\lambda_1}(A), A \cup B \subset f_{\lambda_1}(B),$$

and such that the orbit of every point in $[0,1] \setminus A \cup B$ for f_{λ_1} is attracted to $\mathcal{O}$. Conclude that the set of points of $[0,1]$ whose orbit by f_{λ_1} is not attracted to $\mathcal{O}$ is equal to

$$K := \{x \in [0,1] : \text{ for every } n \geq 0, \ f_{\lambda_1}^n(x) \in A \cup B\}.$$

D. Consider the symbolic space

$$\Sigma := \{(\theta_j)_{j \in \mathbb{N}_0} \in \{A, B\}^{\mathbb{N}_0} : a_j = A \Rightarrow a_{j+1} = B\},$$

the shift map

$$\begin{array}{rccl} \sigma : & \Sigma & \to & \Sigma \\ & (\theta_j)_{j \in \mathbb{N}_0} & \mapsto & \sigma\left((\theta_j)_{j \in \mathbb{N}_0}\right) := (\theta_{j+1})_{j \in \mathbb{N}_0}, \end{array}$$

and the itinerary map $\iota : K \to \Sigma$ defined for every x in K and j in $\mathbb{N}_0$ by

$$\iota(x)_j := \begin{cases} A & \text{if } f_{\lambda_1}^j(x) \in A; \\ B & \text{if } f_{\lambda_1}^j(x) \in B. \end{cases}$$

Show that ι is surjective and that $f_{\lambda_1} \circ \iota = \iota \circ \sigma$.

E. Assuming that there is a smooth metric on $A \cup B$ for which f_{λ_1} is uniformly expanding on K, prove that K has zero Lebesgue measure and that ι is a homeomorphism.

Exercise 1.5 (Ulam and von Neumann map).

A. Prove (1.3).

B. Let $h : [0,1] \to [-1,1]$ be defined by $h(x) := 1 - 2x$, and prove that $h \circ f_4 \circ h^{-1}(y)$ is equal to the degree 2 Chebyshev polynomial $C(y) := 2y^2 - 1$.

C. Using the identity $C(\cos(\theta)) = \cos(2\theta)$, and that the angle doubling map m_2 is ergodic with respect to Leb (Exercise 1.2), prove that f_4 is ergodic with respect to ν, and with respect to the Lebesgue measure on $[0,1]$.

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2. Uniformly expanding maps

The purpose of this section[†] is to workout in full details a result Keller on the existence and statistical properties of acip's for (piecewise) expanding interval maps [Kel85, Theorems 3.2 and 3.3]. We stress that in this result the maps are not assumed to be Markovian, *cf.* Figure 2.1. Keller's result derives a very strong (technical) conclusion: The transfer operator acts with a spectral gap on a certain (carefully chosen) Banach space. This easily implies the existence of an acip with strong stochastic properties, see for example Sarig's notes on the II Brazilian School on Dynamical Systems.

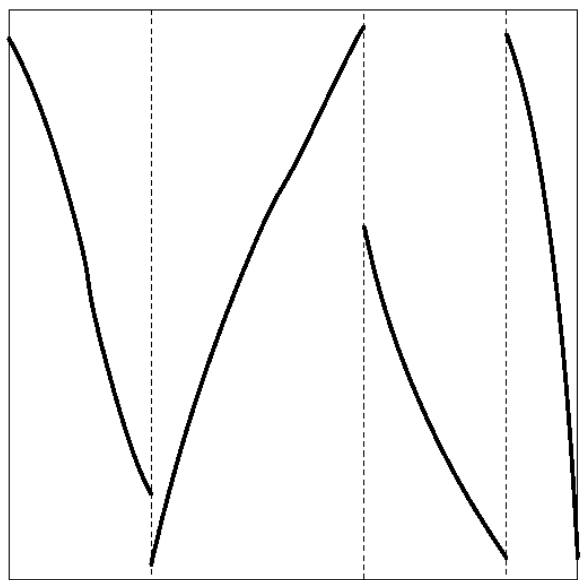

Figure 2.1. Graph of a piecewise expanding interval map.

One of the main features of Keller's result is that it is sufficiently general to apply to maps whose derivative is Hölder continuous. Arguably, the main contribution of Keller in [Kel85] is the introduction of a Banach space of functions with the following key properties:

- It contains the space of Hölder continuous functions of a fixed exponent;
- Under very natural assumptions the transfer operator acts on this space with a spectral gap.

Note that the transfer operator of an interval map does not preserve the space of continuous functions (except in very special cases, see Exercise 2.2), so any function space satisfying the properties above must contain discontinuous functions. Roughly speaking, the function space introduced by Keller is formed by functions having a local Hölder constant at almost every point,

[†]Written in collaboration with Huaibin Li.

in such a way that this local Hölder constant is integrable as a function of the point. See §2.2.1 for the precise definition of the function spaces introduced by Keller, and §2.3 for its relations to other function spaces.

The classical result of Lasota and Yorke [LY73] also gives the spectral gap property, but it requires a higher regularity of the interval map: Its derivative should be of bounded variation. Although this is enough for many applications, it does leave out some important maps, like the one-dimensional reduction of the Lorenz flow. In fact, such a map is Hölder continuous but it is usually not of bounded variation. Another setting where the hypotheses of the result of Lasota and Yorke are too restrictive is in the thermodynamic formalism of smooth maps: In the classical setting the potential function is only assumed to be Hölder continuous. In §2.1 we briefly state an application in this direction that illustrates the power of Keller's result.

The rest of this section is organized as follows. In §2.2 we introduce Keller's function spaces (§2.2.1) and state Keller's spectral gap theorem (Theorem 6 in §2.2.2). We also state a consequence of Keller's result for topologically exact maps (Corollary 2.4). In §2.3 we relate Keller's spaces to some well-known function spaces. The proof of Theorem 6 is given in §2.4. As in other results of this type, the main ingredient is a "2 norms inequality" that we sate as Proposition 2.9. The proof of Theorem 6 form this result is a relatively standard application of an ergodic theorem of Ionescu-Tulcea and Marinescu [ITM50]. The proof of Corollary 2.4 is given in §2.5.

There is a list of exercises at the end of this section (§2.6).

2.1. **Equilibrium states and the absence of phase transitions.** We start recalling a few concepts of thermodynamic formalism, see for example [Kel98] or [PU10] for background. Let (X, dist) be a compact metric space, and let $T : X \to X$ be a continuous map. Denote by $\mathcal{M}(X)$ the space of Borel probability measures on X endowed with the weak* topology, and by $\mathcal{M}(X, T)$ the subspace of $\mathcal{M}(X)$ of those measures that are invariant by T. For each measure ν in $\mathcal{M}(X, T)$, denote by $h_\nu(T)$ the *measure-theoretic entropy* of ν. For a continuous function $\phi : X \to \mathbb{R}$, denote by $P(X, \phi)$ the *topological pressure of T for the potential ϕ*, defined by

$$P(T, \phi) := \sup \left\{ h_\nu(T) + \int_X \phi \, d\nu : \nu \in \mathcal{M}(X, T) \right\}. \tag{2.1}$$

An *equilibrium state of T for the potential ϕ* is a measure at which the supremum above is attained.

Let I be a compact interval in $\mathbb{R}$. A continuous map $f : I \to I$ is *multimodal* if it is not injective, and if there is a finite partition of I into intervals on each of which f is injective. When f is differentiable, a point of I is *critical for f* if the derivative of f vanishes at it. We denote by $\mathrm{Crit}(f)$ the set of critical points of f. In what follows we denote by $\mathscr{A}$ the collection of all those differentiable multimodal maps f such that:

- Df is Hölder continuous;
- $\mathrm{Crit}(f)$ is finite;

The following is a key hypothesis:

Definition 2.1. Let $f : I \to I$ be an interval map in $\mathscr{A}$. Then a continuous potential $\varphi : I \to \mathbb{R}$ is *hyperbolic for* f, if for some integer $n \geq 1$, the function $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ f^j$ satisfies

$$\sup_I \frac{1}{n} S_n(\varphi) < P(f, \varphi).$$

Note that the potential $\varphi = -\log |Df|$ is hyperbolic for f if and only if f is uniformly expanding.

Theorem 5 ([LRL14a], Corollary 1.4)**.** *Let I be a compact interval of $\mathbb{R}$ and let $f : I \to I$ be an interval map in $\mathscr{A}$ that is topologically exact on I. Then for every Hölder continuous potential $\varphi : I \to \mathbb{R}$ that is hyperbolic for f, there is a unique equilibrium state ν of f for the potential φ and the measure-theoretic entropy of this measure is strictly positive. Moreover, this measure is exponentially mixing and the pressure function is real analytic.*

This result is obtained by first constructing a conformal measure using the Patterson-Sullivan method,[‡] and then applying Keller's result. This approach turns out to be more efficient than the inducing scheme approach, used for example in [BT08], as it does not rely on any bounded distortion hypothesis, and it applies to a larger class of maps.

Under a weak non-uniform hyperoblicity assumption on the map f, it can be shown that the hyperbolicity hypothesis is satisfied for every Hölder continuous potential. The following is a sample result: For a map f as in Theorem 5, every Hölder continuous potential φ is hyperbolic for f, provided that f satisfies some additional regularity assumptions, that all the periodic points of f are hyperbolic repelling, and that for every critical value v of f we have

$$\lim_{n \to +\infty} |Df^n(v)| = +\infty,$$

see [LRL14b, Theorem A], and also [IRRL12] for an earlier result in the complex setting. Recall that a periodic point p of f of period n is *hyperbolic repelling*, if $|Df^n(p)| > 1$. Thus, for such a map f the conclusions of Theorem 5 hold for all Hölder continuous functions φ. In particular, the pressure function is real analytic, so there are no "phase transitions".

For interval maps and bounded variation potentials, the hyperbolicity hypothesis appears naturally in various results, see for example [BK90, DKU90, HK82, Kel85, LSV98, Rue94] and references therein, as well as Baladi's book [Bal00, §3].

[‡]To be able to apply the Patterson-Sullivan construction some *a priori* estimates on the transfer operator are needed, and this requires a bit of Pesin theory. This seems to be the only part in the proof where the derivative of f is required to be Hölder continuous.

For complex rational maps and Hölder continuous potentials, the hyperbolicity hypothesis also appears naturally in [Hay99, DPU96, DU91, Prz90] where the results of Freire, Lopes, and Mañé [FLM83, Mañ83] and Ljubich [Lju83] for the maximal entropy measure were extended to the thermodynamic formalism. In this approach the results are obtained by showing directly that the transfer operator is quasi-periodic. This approach gives a good understanding of equilibrium states and their statistical properties, although the method is rather involved. A more conceptual approach was taken in [SUZ11]: A Young tower was constructed, and for the first time the absence of phase transitions was obtained.

It is not known whether these results (in the complex setting) could be obtained by the transfer operator method. The main problem is to find a function space that includes Hölder continuous functions (of a given exponent) and on which the transfer operator acts nicely. This is formulated more precisely in the following problem.

Problem 2.2. *Given a complex rational map f and a Hölder continuous potential φ, find a function space containing the space of Hölder continuous functions, and on which the transfer operator of f with potential φ acts with a spectral gap.*

2.2. **Keller's spaces and spectral gap theorem.**

2.2.1. *Keller's function spaces.* Let X be a compact subset of $\mathbb{R}$, and m a Borel non-atomic probability measure on X. We consider the equivalence relation on the space of complex valued functions defined on X, defined by agreement on a set of full measure with respect to m. Denote by d the pseudo-distance on X defined by

$$d(x,y) := m(\{z \in X : x \leq z \leq y \text{ or } y \leq z \leq x\}).$$

Note that for every x in X and every $\varepsilon > 0$, the set

$$B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$$

has strictly positive measure with respect to m.

Given a measurable function $h : X \to \mathbb{C}$ and $\varepsilon > 0$, for each $x \in X$ put

$$\operatorname{osc}(h,\varepsilon,x) := \text{ess-sup}\{|h(y) - h(y')| : y, y' \in B(x,\varepsilon)\}$$

and

$$\operatorname{osc}_1(h,\varepsilon) := \int_X \operatorname{osc}(h,\varepsilon,x)\ dm(x).$$

Fix $A > 0$, and for each $\alpha \in (0,1]$ and each $h : X \to \mathbb{C}$ put

$$\operatorname{Var}_{\alpha,1}(h) := \sup_{\varepsilon \in (0,A]} \frac{\operatorname{osc}_1(h,\varepsilon)}{\varepsilon^\alpha} \text{ and } \|h\|_{\alpha,1} := \|h\|_1 + \operatorname{Var}_{\alpha,1}(h).$$

Note that $\operatorname{Var}_{\alpha,1}(h)$ and $\|h\|_{\alpha,1}$ only depend on the equivalence class of h. Let $\mathrm{H}^{\alpha,1}(m)$ be the space of equivalence classes of functions $h : X \to \mathbb{C}$, such that $\|h\|_{\alpha,1} < +\infty$. Note that $\mathrm{Var}_{\alpha,1}(\cdot)$ and $\|\cdot\|_{\alpha,1}$ induce a semi-norm and a norm on $\mathrm{H}^{\alpha,1}(m)$, respectively; by abuse of notation we denote these functions also by $\mathrm{Var}_{\alpha,1}(\cdot)$ and $\|\cdot\|_{\alpha,1}$.

2.2.2. *Spectral gap theorem.* Keller's theorem is stated in a general setting that includes that of piecewise expanding maps as a special case, see Exercise 2.3.

The map: Let I be a compact interval of $\mathbb{R}$, $N \geq 2$ an integer, and $\mathscr{P}' := \{I_1', \cdots, I_N'\}$ a partition of I into intervals. Let $T : I \to I$ be a transformation on I that is continuous and monotone on each I_i' in $\mathscr{P}'$. Furthermore, let X be a compact subset of I such that $T^{-1}(X) = X$, for each i in $\{1, \cdots, N\}$ put $I_i := I_i' \cap X$, and put $\mathscr{P} := \{I_1, \cdots, I_N\}$.

The potential: Fix $p \geq 1$. A function $h : X \to \mathbb{C}$ is of *bounded p-variation*, if

$$\sup \left\{ \left(\sum_{i=1}^{k} |h(x_i) - h(x_{i-1})|^p \right)^{1/p} : k \geq 1, x_0, \ldots, x_k \in X, x_0 < \cdots < x_k \right\}$$

is finite.

Let $g : X \to [0, +\infty)$ be a function of bounded p-variation, and let $\mathscr{L}_g$ be the operator acting on the space

$$\mathrm{Eb}(X) := \{h : X \to \mathbb{C} \text{ measurable and bounded in absolute value}\},$$

defined by

$$\mathscr{L}_g(h)(x) := \sum_{y \in T^{-1}(x)} h(y)g(y) = \sum_{i \in \{1,\ldots,N\}, x \in T(I_i)} (h \cdot g) \circ T|_{I_i}^{-1}(x). \tag{2.2}$$

Remark 2.3. The operator $\mathscr{L}_g$ is usually known as the "transfer" or "Ruelle-Perron-Frobenius" operator for the "potential" g. In the case T is differentiable and $g = 1/|DT|$, the operator $\mathscr{L}_g$ describes the action of T on on densities, see Exercise 2.1.

The conformal measure: Assume in addition that there is an atom-free Borel probability measure m on X such that the following properties hold:

- H1. For each I_i in $\mathscr{P}$, the map $T|_{I_i}^{-1}$ is non-singular with respect to m, so that for every subset E of I_i of measure zero, the set $\left(T|_{I_i}^{-1}\right)^{-1}(E) = T(E)$ is also of measure zero;
- H2. On a set of full measure with respect to m, we have

$$g^{-1} = \sum_{i=1}^{N} \frac{d\left(T|_{I_i}^{-1}\right)_* m}{dm};$$

H3. For each h in $\mathrm{Eb}(X)$ we have $\int_X \mathscr{L}_g(h)\ dm = \int_X h\ dm$, and $\mathscr{L}_g$ extends to a positive linear map from $\mathrm{L}^1(m)$ to itself satisfying $\|\mathscr{L}_g(h)\|_1 \leq \|h\|_1$.

In the following theorem, we gather several results from [Kel85].

Theorem 6 ([Kel85], Theorems 3.2 and 3.3)**.** *Let T, g, $\mathscr{L}_g$, and m be as above, and assume that there is an integer $n \geq 1$ such that the function*

$$g_n := g \cdot g \circ T \cdot \cdots \cdot g \circ T^{n-1}$$

satisfies $\sup_X g_n < 1$. Then there is an integer $k \geq 1$ and constants $A > 0$, β in $(0,1)$, and $C > 0$, such that for every function h in $\mathrm{H}^{1/p,1}(m)$, we have

$$\|\mathscr{L}_g^k(h)\|_{1/p,1} \leq \beta \|h\|_{1/p,1} + C\|h\|_1. \tag{2.3}$$

Moreover, the following properties hold:

1. *The set $\mathscr{E}$ of eigenvalues of $\mathscr{L}_g|_{\mathrm{L}^1(m)}$ of modulus 1 is finite. Moreover, for each λ in $\mathscr{E}$, the space*

   $$\mathrm{E}(\lambda) := \{h \in \mathrm{L}^1(m) : \mathscr{L}_g(h) = \lambda h\}$$

   is contained in $\mathrm{H}^{1/p,1}(m)$ and it is of finite dimension;
2. *If for each λ in $\mathscr{E}$ we denote by $\mathscr{P}(\lambda)$ the projection in $\mathrm{L}^1(m)$ to $\mathrm{E}(\lambda)$, then the operator*

   $$\mathscr{Q} := \mathscr{L}_g - \sum_{\lambda \in \mathscr{E}} \lambda \mathscr{P}(\lambda)$$

   maps $\mathrm{H}^{1/p,1}(m)$ to itself, and there is ρ in $(0,1)$ and a constant $M > 0$ such that for every integer $n \geq 0$ we have $\|\mathscr{Q}^n\|_{\alpha,1} \leq M\rho^n$. Finally, for each λ in $\mathscr{E}$ the operators $\mathscr{Q}\mathscr{P}(\lambda)$ and $\mathscr{P}(\lambda)\mathscr{Q}$ are both identically zero, and for each λ' in $\mathscr{E}$ different from λ the operators $\mathscr{P}(\lambda)\mathscr{P}(\lambda')$ and $\mathscr{P}(\lambda')\mathscr{P}(\lambda)$ are also identically zero. In particular, for every integer $n \geq 1$ we have

   $$\mathscr{L}_g^n = \sum_{\lambda \in \mathscr{E}} \lambda^n \mathscr{P}(\lambda) + \mathscr{Q}^n.$$
3. *The set $\mathscr{E}$ contains 1, and if we put $h := \mathscr{P}(1)(\mathbb{1})$, then $\nu := hm$ is a probability measure that is invariant by T.*

The following corollary follows from the previous theorem using known arguments.

Corollary 2.4. *Under the assumptions of Theorem 6, and assuming in addition that T is topologically exact on X, we have the following properties:*

1. *The number 1 is an eigenvalue of $\mathscr{L}_g$ of algebraic multiplicity 1. Moreover, there is ρ in $(0,1)$ such that the spectrum of $\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)}$ is contained in $B(0,\rho) \cup \{1\}$.*

2. *There is a constant $C > 0$ such that for every bounded measurable function $\phi : X \to \mathbb{C}$, and every function ψ in $\mathrm{H}^{1/p,1}(m)$, the measure ν given by part 3 of Theorem 6 satisfies for every integer $n \geq 1$ that*
$$\left| \int \phi \circ T^n \cdot \psi \, d\nu - \int \phi \, d\nu \int \psi \, d\nu \right| \leq C \|\phi\|_\infty \|\psi\|_{1/p,1} \rho^n.$$
3. *Given ψ in $\mathrm{H}^{1/p,1}(m)$, for each τ in $\mathbb{C}$ the operator $\mathscr{L}_\tau$ defined by*
$$\mathscr{L}_\tau(h) := \mathscr{L}_g \left(\exp(\tau\psi) \cdot h \right)$$
maps $\mathrm{H}^{1/p,1}(m)$ to itself and the restriction $\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is bounded. Moreover, $\tau \mapsto \mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is analytic in the sense of Kato on $\mathbb{C}$, and the spectral radius of $\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ depends on a real analytic way on τ on a neighborhood of $\tau = 0$.

After some general considerations on Keller's spaces in §2.3, the proof of Theorem 6 is given in §2.4 and that of Corollary 2.4 in §2.5.

2.3. **Relations to other function spaces.** In this section we relate Keller's spaces to other function spaces. Throughout this section we fix a compact subset X of $\mathbb{R}$.

Given $p \geq 1$ and a function $h : X \to \mathbb{C}$, put

$$\mathrm{Var}_p(h) := \sup \left\{ \left(\sum_{i=1}^{k} |h(x_i) - h(x_{i-1})|^p \right)^{1/p} : k \geq 1, x_0, \ldots, x_k \in X, x_0 < \cdots < x_k \right\}$$

and
$$\|h\|_{\mathrm{BV}_p} := \mathrm{Var}_p(h) + \|h\|_\infty.$$
Recall that h is of bounded p-variation if $\mathrm{Var}_p(h) < +\infty$. Let BV_p be the space of all bounded p-variation functions defined on X. Then $\| \cdot \|_{\mathrm{BV}_p}$ is a norm on BV_p, for which BV_p is a Banach space.

Given α in $(0, 1]$, denote by H^α the space of Hölder continuous functions of exponent α defined on X and taking values in $\mathbb{C}$. For each function $h : X \to \mathbb{C}$ in H^α, put

$$|h|_\alpha := \sup_{x, x' \in X, x \neq x'} \frac{|h(x) - h(x')|}{|x - x'|^\alpha} \text{ and } \|h\|_\alpha := \|h\|_\infty + |h|_\alpha.$$

Then $\| \cdot \|_\alpha$ is a norm on H^α and $(\mathrm{H}^\alpha, \| \cdot \|_\alpha)$ is a Banach space.

Note that these definitions immediately imply that for each h in H^α, we have
$$\mathrm{Var}_{1/\alpha}(h) \leq |\sup X - \inf X|^\alpha \, |h|_\alpha,$$
and therefore
$$\|h\|_{\mathrm{BV}_{1/\alpha}} \leq \max \left\{ 1, |\sup X - \inf X|^\alpha \right\} \|h\|_\alpha. \tag{2.4}$$

In the following proposition we summarize various results from [Kel85, Theorem 1.13].

Proposition 2.5. *Fix $A > 0$, let X be a compact subset of $\mathbb{R}$, and let m be an atom-free Borel probability measure on X. Then for each α in $(0, 1]$, the space $\mathrm{H}^{\alpha,1}(m)$ defined in §2.2.1 satisfies the following properties:*

1. *$(\mathrm{H}^{\alpha,1}(m), \|\cdot\|_{\alpha,1})$ is a Banach space;*
2. *For each $C > 0$, the set*

$$\left\{ f \in \mathrm{H}^{\alpha,1}(m) : \|f\|_{\alpha,1} \leq C \right\} \tag{2.5}$$

   is a compact subset of $\mathrm{L}^1(m)$.

3. *For each function $h : X \to \mathbb{C}$ in $\mathrm{BV}_{1/\alpha}$, we have $\|h\|_{\alpha,1} \leq 2^\alpha \|h\|_{\mathrm{BV}_{1/\alpha}}$;*
4. *Each function in $\mathrm{H}^{\alpha,1}(m)$ is essentially bounded. In fact, there is a constant $C_* > 0$ such that each element h of $\mathrm{H}^{\alpha,1}(m)$ satisfies $\|h\|_\infty \leq C_* \|h\|_{\alpha,1}$.*

Note that by combining (2.4) with parts 3 and 4 of the proposition above, we obtain

$$\mathrm{H}^\alpha \subset \mathrm{BV}_{1/\alpha} \subset \mathrm{H}^{\alpha,1}(m) \subset \mathrm{L}^\infty(m).$$

We conclude that each of the spaces $\mathrm{BV}_{1/\alpha}$ and $\mathrm{H}^{\alpha,1}(m)$ is dense in $\mathrm{L}^1(m)$.

The rest of this section is devoted to the proof of Proposition 2.5.

Lemma 2.6. *For every $p \geq 1$ and $\varepsilon > 0$, we have for every measurable function $h : X \to \mathbb{R}$*

$$\int_X \mathrm{osc}(h, \varepsilon, x)^p \, dm(x) \leq 2\varepsilon \, \mathrm{Var}_p(h)^p.$$

Proof. Put $a := \inf X$ and for each t in $[0, m(X)]$, put

$$x(t) := \sup\{x \in X : d(x, a) = t\}.$$

Suppose first $\varepsilon \geq m(X)/2$. Using that for every x in X we have $\mathrm{osc}(h, \varepsilon, x)^p \leq \mathrm{Var}_p(h)^p$, we obtain

$$\int_X \mathrm{osc}(h, \varepsilon, x)^p \, dm(x) \leq m(X) \, \mathrm{Var}_p(h)^p \leq 2\varepsilon \, \mathrm{Var}_p(h)^p.$$

It remains to consider the case $\varepsilon < m(X)/2$. For every ξ in $[0, 2\varepsilon]$, put

$$n_\varepsilon(\xi) := \max\{ \text{ nonnegative integer } n : \xi + 2n\varepsilon \leq m(X)\}.$$

Since the balls $(B_d(x(\xi + 2k\varepsilon), \varepsilon))_{k=0}^{n_\varepsilon(\xi)}$ are pairwise disjoint, we have

$$\sum_{k=0}^{n_\varepsilon(\xi)} \mathrm{osc}\left(h, \varepsilon, x(\xi + 2k\varepsilon)\right)^p \leq \mathrm{Var}_p(h)^p.$$

It follows that

$$\int_X \mathrm{osc}(h, \varepsilon, x)^p \, dm(x) = \int_0^{2\varepsilon} \sum_{k=0}^{n_\varepsilon(\xi)} \mathrm{osc}(h, \varepsilon, x(\xi + 2k\varepsilon'))^p \, d\xi \leq 2\varepsilon \, \mathrm{Var}_p(h)^p,$$

and so we obtain the lemma. □

Lemma 2.7. *For every bounded measurable function $h : X \to \mathbb{C}$ and every ε in $(0, m(X)]$, we have*

$$\|h\|_\infty \le \frac{1}{\varepsilon} \operatorname{osc}_1(h, \varepsilon) + \frac{1}{m(X)} \left| \int h \, dm \right|$$

Proof. Putting

$$h_0 := \frac{1}{m(X)} \int h \, dm, g := h - h_0,$$

we have

$$\int g(x) \, dm(x) = 0 \text{ and } \|h\|_\infty = \|(h - h_0) + h_0\|_\infty \le \|g\|_\infty + |h_0|.$$

Since for every $x \in X$ we have $\operatorname{osc}(h, \varepsilon, x) = \operatorname{osc}(g, \varepsilon, x)$, it suffices to prove that $\|g\|_\infty \le \frac{1}{\varepsilon} \operatorname{osc}_1(g, \varepsilon)$. Replacing g by $-g$ if necessary, assume

$$\|g\|_\infty = \text{ess-sup}_X \, g.$$

Since $\int g(y) \, dm(y) = 0$, this implies that $\text{ess-inf}_X \, g \le 0$.

For each t in $[0, m(X)]$ put

$$x(t) := \sup\{x \in X : d(x, \inf X) \le t\},$$

and for each ξ in $[0, \varepsilon)$ put

$$n(\xi) := \{\text{nonnegative integer } n : \xi + \varepsilon n \le m(X)\}.$$

Then we have

$$X \subset \bigcup_{k=0}^{n(\xi)} B(x(\xi + k\varepsilon), \varepsilon).$$

It follows that there are k_1 and k_2 in $\{0, 1, \ldots, n(\xi)\}$ such that

$$\text{ess-sup}_{B(x(\xi + k_1\varepsilon), \varepsilon)} \, g = \|g\|_\infty \text{ and } \text{ess-inf}_{B(x(\xi + k_2\varepsilon), \varepsilon)} \, g \le 0,$$

so

$$\|g\|_\infty \le \text{ess-sup}_X \, g - \text{ess-inf}_X \, g \le \sum_{k=0}^{n(\xi)} \operatorname{osc}\left(g, \varepsilon, x(\xi + k\varepsilon)\right).$$

Since this holds for every ξ in $[0, \varepsilon)$, we have

$$\begin{aligned} \|g\|_\infty &\le \frac{1}{\varepsilon} \int_0^\varepsilon \sum_{k=0}^{n(\xi)} \operatorname{osc}\left(g, \varepsilon, x(\xi + 2k\varepsilon)\right) \, d\xi \\ &= \frac{1}{\varepsilon} \int \operatorname{osc}\left(g, \varepsilon, y\right) \, dm(y). \end{aligned}$$

This implies completes the proof of the lemma. □

Lemma 2.8. *Let α be in $(0, 1]$, and let $(h_n)_{n=1}^{+\infty}$ be a sequence in $\mathrm{H}^{\alpha,1}(m)$. If there is h in $\mathrm{L}^1(m)$ such that $(h_n)_{n=1}^{+\infty}$ converges to h in $\mathrm{L}^1(m)$, then*

$$|h|_{\alpha,1} \le \liminf_{n \to +\infty} |h_n|_{\alpha,1}.$$

Proof. It is enough to show that for every ε in $(0, A]$ and every x in X we have

$$\operatorname{osc}(h, \varepsilon, x) \le \liminf_{n \to +\infty} \operatorname{osc}(h_n, \varepsilon, x).$$

Let $\Delta : B(x, \varepsilon) \times B(x, \varepsilon) \to \mathbb{R}$, and for each n let $\Delta_n : B(x, \varepsilon) \times B(x, \varepsilon) \to \mathbb{R}$ be defined by

$$\Delta(x', x'') := h(x') - h(x'') \text{ and } \Delta_n(x', x'') := h_n(x') - h_n(x'').$$

Then $(\Delta_n)_{n=1}^{+\infty}$ converges to Δ in $L^1\left(\left(m|_{B(x,\varepsilon)}\right)^2\right)$, and therefore

$$\operatorname{osc}(h, \varepsilon, x) = \operatorname{ess-sup} |\Delta| \le \liminf_{n \to +\infty} \operatorname{ess-sup} |\Delta_n| = \liminf_{n \to +\infty} \operatorname{osc}(h_n, \varepsilon, x).$$

□

Proof of Proposition 2.5. Part 4 with $C_* = \min\{m(X)^{-1}, A^{-(1-\alpha)}, m(X)^{-(1-\alpha)}\}$ is a direct consequence of Lemma 2.7 with $\varepsilon = \min\{A, m(X)\}$.

To prove part 1 it suffices to check that $\mathrm{H}^{\alpha,1}(m)$ is complete with respect to $\| \cdot \|_{\alpha,1}$. Let $(h_n)_{n=1}^{+\infty}$ be a Cauchy sequence in $\mathrm{H}^{\alpha,1}(m)$. Then $(h_n)_{n=1}^{+\infty}$ is also a Cauchy sequence in $\mathrm{L}^1(m)$, and therefore there is h in $\mathrm{L}^1(m)$ such that $\|h_n - h\|_1 \to 0$ as $n \to +\infty$. By Lemma 2.8, we have

$$|h|_{\alpha,1} \le \liminf_{n \to +\infty} |h_n|_{\alpha,1} < +\infty.$$

It follows that h is in $\mathrm{H}^{\alpha,1}(m)$. To complete the proof of part 1, it is enough to prove that for every $\delta > 0$ there is $N > 0$ such that for each integer $n \ge N$ we have

$$|h_n - h|_{\alpha,1} \le \delta.$$

In fact, since $(h_n)_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathrm{H}^{\alpha,1}(m)$, there is $N > 0$ such that for each pair of integers $k, n \ge N$, we have $\|h_k - h_n\|_{\alpha,1} < \delta$. Fix $n \ge N$, and note that $h - h_k + (h_n - h)$ converges to $h_n - h$ in $\mathrm{L}^1(m)$ as $k \to +\infty$. It follows from Lemma 2.8 again that

$$|h_n - h|_{\alpha,1} \le \liminf_{k \to +\infty} |h - h_k + (h_n - h))|_{\alpha,1} = \liminf_{k \to +\infty} |h_n - h_k|_{\alpha,1} \le \delta.$$

The proof of part 1 is complete.

Since $\mathrm{L}^1(m)$ is complete, to prove part 2 it is enough to show that for every $\varepsilon > 0$ there is a finite subset of (2.5) that is ε-dense with respect to $\| \cdot \|_1$. To do this, let $\mathscr{P}$ be a partition of X such that

$$|\mathscr{P}| := \max_{P \in \mathscr{P}} \sup \left\{ d(x, x') : x, x' \in P \right\}$$

satisfies

$$0 < |\mathscr{P}| \le \min \left\{ A, (\varepsilon/(3C))^{1/\alpha} \right\}.$$

Then for each h in $\mathrm{H}^{\alpha,1}(m)$ and each P in $\mathscr{P}$, the number

$$h_P := \frac{1}{m(P)} \int_P h \, dm$$

satisfies

$$\text{ess-inf}\,\{h(x) : x \in P\} \leq h_P \leq \text{ess-sup}\,\{h(x) : x \in P\}.$$

It follows that

$$\text{ess-sup}\,\{|h(x) - h_P| : x \in P\} \leq \text{ess-sup}\,\left\{|h(x) - h(x')| : x, x' \in P\right\},$$

and therefore that the function

$$h_{\mathscr{P}} := \sum_{P \in \mathscr{P}} h_P \cdot \mathbb{1}_P$$

satisfies

$$\|h - h_{\mathscr{P}}\|_1 \leq \operatorname{osc}_1(h - h_{\mathscr{P}}, |\mathscr{P}|) \leq |\mathscr{P}|^\alpha \operatorname{Var}_{\alpha,1}(h). \tag{2.6}$$

Thus, the image of (2.5) by the linear map $L : \mathrm{H}^{\alpha,1}(m) \to \mathbb{R}^{\mathscr{P}}$ defined by

$$L(h) = (h_P)_{P \in \mathscr{P}},$$

is bounded in $\mathbb{R}^{\mathscr{P}}$ with respect to the maximum norm $\| \cdot \|_{\mathscr{P}}$. It follows that there is a finite subset F of (2.5) such that, if we put $\varepsilon' := |\mathscr{P}|\varepsilon/(3\#\mathscr{P})$, then $L(F)$ is ε'-dense in the image of (2.5) by L. To prove that F is ε-dense in (2.5) with respect to $\| \cdot \|_1$, let h be a function in this set and let $\widetilde{h}$ in F be such that $\left\|L(h) - L(\widetilde{h})\right\|_{\mathscr{P}} \leq |\mathscr{P}|\varepsilon/3$. Using (2.6) twice, we obtain

$$\begin{aligned}\left\|h - \widetilde{h}\right\|_1 &\leq \|h - h_{\mathscr{P}}\|_1 + \left\|h_{\mathscr{P}} - \widetilde{h}_{\mathscr{P}}\right\|_1 + \left\|\widetilde{h} - \widetilde{h}_{\mathscr{P}}\right\|_1 \\ &\leq 2\varepsilon/3 + \left\|L(h) - L(\widetilde{h})\right\|_{\mathscr{P}} \sum_{P \in \mathscr{P}} \frac{1}{m(P)} \leq \varepsilon.\end{aligned}$$

This completes the proof of part 2.

It remains to prove 3. By Hölder's integral inequality and Lemma 2.6 with $p = 1/\alpha$, for every ε in $(0, A]$ we have

$$\operatorname{osc}_1(h, \varepsilon) \leq \left(\int_X \operatorname{osc}(h, \varepsilon, x)^{1/\alpha} \, dm(x)\right)^\alpha \leq 2^\alpha \varepsilon^\alpha \operatorname{Var}_{1/\alpha}(h).$$

It follows that $|h|_{\alpha,1} \leq 2^\alpha \operatorname{Var}_{1/\alpha}(h)$. On the other hand, since $\|h\|_1 \leq \|h\|_\infty$, we have

$$\|h\|_{\alpha,1} = \|h\|_1 + |h|_{\alpha,1} \leq \|h\|_\infty + 2^\alpha \operatorname{Var}_{1/\alpha}(h) \leq 2^\alpha \|h\|_{\mathrm{BV}_{1/\alpha}}.$$

□

2.4. **The 2 norms inequality.** This section is devoted to the proof of Theorem 6. We follow [Kel85], though some places are streamlined. The following is the main ingredient in the proof, which is usually known as the "2 norms inequality".

Proposition 2.9 (2 norms inequality)**.** *Let T, g, and $\mathscr{L}_g$ be as in §2.2.2, and assume that in addition we have $\gamma := \sup_X g < 1$. Then for each $\delta > 0$, there are $A > 0$ and $K > 0$ such that for each function h in $\mathrm{H}^{1/p,1}(m)$ we have*

$$\mathrm{Var}_{1/p,1}(\mathscr{L}_g(h)) \leq (6+\delta)\gamma^{1/p}\,\mathrm{Var}_{1/p,1}(h) + K\|h\|_1.$$

The proof of Theorem 6 form this result is a relatively standard application of an ergodic theorem of Ionescu-Tulcea and Marinescu [ITM50]. It is given at the end of this section. The proof of the proposition above is given after a couple of lemmas.

Throughout the rest of this section we fix X and m as in §2.2.1.

Lemma 2.10. *Let I' be an interval of $\mathbb{R}$ such that $X' := X \cap I'$ satisfies $m(X') > 0$. Then for every bounded measurable function $h : X \to \mathbb{C}$ and every ε in $(0, m(X')/2]$, we have*

$$\mathrm{osc}_1\left(h \cdot \mathbb{1}_{X'}, \varepsilon\right) \leq 6 \int_{X'} \mathrm{osc}\left(h|_{X'}, \varepsilon, x\right)\ dm(x) + 4\varepsilon \frac{1}{m(X')}\left|\int_{X'} h\ dm\right|.$$

Proof. Replacing I' by its closure if necessary, assume I' is compact. It follows that X' is also compact.

Putting

$$h_0 := \int_{X'} h\ dm \text{ and } g := h - h_0,$$

we have

$$\begin{aligned}\mathrm{osc}_1(h \cdot \mathbb{1}_{X'}, \varepsilon) &\leq \mathrm{osc}_1(g \cdot \mathbb{1}_{X'}, \varepsilon) + \mathrm{osc}_1(h_0 \cdot \mathbb{1}_{X'}, \varepsilon)\\ &\leq \mathrm{osc}_1(g \cdot \mathbb{1}_{X'}, \varepsilon) + 4\varepsilon|h_0|.\end{aligned}$$

So, it is enough to show that

$$\mathrm{osc}_1(g \cdot \mathbb{1}_{X'}, \varepsilon) \leq 6 \int_{X'} \mathrm{osc}(g|_{X'}, \varepsilon, x)\ dm(x).$$

For x in X that is not necessarily in X', put

$$\mathrm{osc}(g|_{X'}, \varepsilon, x) := \text{ess-sup}\{|g(y) - g(y')| : y, y' \in B(x, \varepsilon) \cap X'\}.$$

Note that when x is in X' this coincides with the definition in §2.2.1. Putting

$$x'_{\inf} := \inf X' \text{ and } x'_{\sup} := \sup X',$$

by Lemma 2.7 with $h = g|_{X'}$ we have

$$\begin{aligned}
\operatorname{osc}_1(g \cdot \mathbb{1}_{X'}, \varepsilon) &= \int_{X' \setminus B(\{x'_{\inf}, x'_{\sup}\}, \varepsilon)} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) \\
&\quad + \int_{B(\{x'_{\inf}, x'_{\sup}\}, \varepsilon)} \operatorname{osc}(g \cdot \mathbb{1}_{X'}, \varepsilon, x) \ dm(x) \\
&\le \int_{X' \setminus B(\{x'_{\inf}, x'_{\sup}\}, \varepsilon)} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) \\
&\quad + \int_{B(\{x'_{\inf}, x'_{\sup}\}, \varepsilon)} \operatorname{osc}(g|_{X'}, \varepsilon, x) + \left\| g|_{B(x, \varepsilon) \cap X'} \right\|_\infty \ dm(x) \\
&\le \int_{B(X', \varepsilon)} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) \\
&\quad + 2\varepsilon \left(\left\| g|_{B(x'_{\inf}, 2\varepsilon) \cap X'} \right\|_\infty + \left\| g|_{B(x'_{\inf}, 2\varepsilon) \cap X'} \right\|_\infty \right) \\
&\le \int_{B(X', \varepsilon)} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) + 4\varepsilon \left\| g|_{X'} \right\|_\infty \\
&\le \int_{B(X', \varepsilon) \setminus X'} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) + 5 \int_{X'} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x).
\end{aligned}$$

For each t in $[0, m(X')]$ put

$$x'(t) := \sup\{x \in X' : d(x, \inf X') \le t\}.$$

Noting that for every x in $B(x'_{\inf}, \varepsilon) \setminus X'$ we have

$$\operatorname{osc}(g|_{X'}, \varepsilon, x) \le \operatorname{osc}(g|_{X'}, \varepsilon, x'(d(x'_{\inf}, x))),$$

and that for every x in $B(x'_{\sup}, \varepsilon) \setminus X'$ we have

$$\operatorname{osc}(g|_{X'}, \varepsilon, x) \le \operatorname{osc}(g|_{X'}, \varepsilon, x'(m(X') - d(x'_{\sup}, x))),$$

we conclude that

$$\begin{aligned}
&\int_{B(X', \varepsilon) \setminus X'} \operatorname{osc}(g|_{X'}, \varepsilon, x) \ dm(x) \\
&\qquad \le \int_0^\varepsilon \operatorname{osc}(g|_{X'}, \varepsilon, x'(t)) \ dt + \int_{m(X') - \varepsilon}^{m(X')} \operatorname{osc}(g|_{X'}, \varepsilon, x'(t)) \ dt \\
&\qquad \le \int_{X'} \operatorname{osc}(g|_{I_*}, \varepsilon, x) \ dm(x).
\end{aligned}$$

Together with the chain of inequalities above, this completes the proof of the lemma. □

Lemma 2.11. *Let $\mathscr{P}$, T, and g be as in §2.2.2, and suppose that in addition we have $\gamma := \sup_X g < 1$. Then for every $\varepsilon > 0$, every I_* in $\mathscr{P}$, and every* *measurable function $h : X \to \mathbb{C}$, we have*

$$\int_{T(I_*)} \operatorname{osc}\left((h \cdot g) \circ T|_{I_*}^{-1}, \varepsilon, z\right) \ dm(z) \leq \int_{I_*} \operatorname{osc}\left(h|_{I_*}, \gamma\varepsilon, y\right) \ dm(y) + 3\|h|_{I_*}\|_\infty \int_{T(I_*)} \operatorname{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z\right) \ dm(z).$$

Proof. Fix z in $T(I_*)$ and put $w := T|_{I_*}^{-1}(z)$. Given z_1 and z_2 in $B(z,\varepsilon)$, if we put $w_1 := T|_{I_*}^{-1}(z_1)$ and $w_2 := T|_{I_*}^{-1}(z_2)$, then we have

$$\begin{aligned}\left|(h \cdot g) \circ T|_{I_*}^{-1}(z_1) - (h \cdot g) \circ T|_{I_*}^{-1}(z_2)\right| &\leq |h(w_1) - h(w_2)| \cdot |g(w)| + |h(w_1) - h(w_2)| \cdot |g(w) - g(w_1)| \\ &\quad + |g(w_1) - g(w_2)| \cdot |h(w_1)|\end{aligned}$$

Thus, for z in a subset of $T(I_*)$ with full measure with respect to m, we have

$$\operatorname{osc}\left((h \cdot g) \circ T|_{I_*}^{-1}, \varepsilon, z\right) \leq \left(g \circ T|_{I_*}^{-1}\right)(z) \cdot \operatorname{osc}\left(h \circ T|_{I_*}^{-1}, \varepsilon, z\right) + 3\left\|h|_{I_*}\right\|_\infty \cdot \operatorname{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z\right).$$

So the lemma follows from,

$$\begin{aligned}\int_{T(I_*)} \left(g \circ T|_{I_*}^{-1}\right)(z) \cdot \operatorname{osc}\left(h \circ T|_{I_*}^{-1}, \varepsilon, z\right) \ dm(z) &\leq \int_{I_*} g(y) \cdot \operatorname{osc}\left(h|_{I_*}, \gamma\varepsilon, y\right) \ d(T|_{I_*}^{-1})_* m(y) \\ &\leq \gamma \int_{I_*} \operatorname{osc}\left(h|_{I_*}, \gamma\varepsilon, y\right) \ d(T|_{I_*}^{-1})_* m(y) \\ &< \int_{I_*} \operatorname{osc}\left(h|_{I_*}, \gamma\varepsilon, y\right) \ dm(y).\end{aligned}$$

□

Proof of Proposition 2.9. Put

$$\delta_\dagger := \frac{(2\gamma)^{1/p}}{72}\delta.$$

Note that each function of bounded p-variation has one-sided limits at each point, and that its set of discontinuities is at most countable. Hence, we can refine $\mathscr{P}$ if necessary to assume that for every I_* in $\mathscr{P}$ we have $\operatorname{Var}_p(g|_{I_*}) \leq \delta_\dagger$, and that

$$\Gamma_- := \min\{m(T(I_*)) : I_* \in \mathscr{P}, m(T(I_*)) > 0\}$$

and

$$\Gamma_+ := \max\{m(T(I_*)) : I_* \in \mathscr{P}\},$$

satisfy $\Gamma_+ \le 2\Gamma_-$.

Put

$$A := \frac{\Gamma_-}{2}, \Lambda := \min\{m(I_*) : I_* \in \mathscr{P}, m(I_*) > 0\},$$

and

$$K := \frac{(4 + \delta\gamma^{1/p})A^{1-1/p}}{\Lambda}.$$

Then for each ε in $(0, A]$ and each h in $\mathrm{H}^{1/p,1}$, we have

$$\begin{aligned}
\mathrm{osc}_1(\mathscr{L}_g(h), \varepsilon) &= \int_X \mathrm{osc}\left(\sum_{I_* \in \mathscr{P}} \left((h \cdot g) \circ T|_{I_*}^{-1} \cdot \mathbb{1}_{T(I_*)} \right), \varepsilon, x \right) \, dm(x) \\
&\le \sum_{I_* \in \mathscr{P}} \int_X \mathrm{osc}\left((h \cdot g) \circ T|_{I_*}^{-1} \cdot \mathbb{1}_{T(I_*)}, \varepsilon, x \right) \, dm(x) \\
&= \sum_{I_* \in \mathscr{P}} \mathrm{osc}_1((h \cdot g) \circ T|_{I_*}^{-1} \cdot \mathbb{1}_{T(I_*)}, \varepsilon).
\end{aligned}$$

Since by definition $\Gamma_- = 2A$, we can apply Lemmas 2.10 and 2.11, and the definition of Λ to obtain

$$\begin{aligned}
(2.7) \quad & \mathrm{osc}_1(\mathscr{L}_g(h), \varepsilon) \\
&\le 6 \sum_{I_* \in \mathscr{P}} \int_{T(I_*)} \mathrm{osc}\left((h \cdot g) \circ T|_{I_*}^{-1}, \varepsilon, x \right) \, dm(x) \\
&\quad + 4\varepsilon \sum_{\substack{I_* \in \mathscr{P} \\ m(I_*) > 0}} \frac{1}{m(I_*)} \left| \int_{I_*} h \, dm \right| \\
&\le 6 \sum_{I_* \in \mathscr{P}} \int_{I_*} \mathrm{osc}\left(h|_{I_*}, \gamma\varepsilon, y \right) \, dm(y) \\
&\quad + 18 \sum_{I_* \in \mathscr{P}} \|h|_{I_*}\|_\infty \int_{T(I_*)} \mathrm{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z \right) \, dm(z) + \frac{4\varepsilon}{\Lambda} \|h\|_1 \\
&\le 6 \, \mathrm{osc}_1\left(h, \gamma\varepsilon \right) + \frac{4\varepsilon}{\Lambda} \|h\|_1 \\
&\quad + 18 \sum_{I_* \in \mathscr{P}} \|h|_{I_*}\|_\infty \int_{T(I_*)} \mathrm{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z \right) \, dm(z).
\end{aligned}$$

On the other hand, putting $q := 1 - 1/p$, by Lemma 2.6 and the definitions of $\delta_\dagger$ and A, we have for each I_* in $\mathscr{P}$

$$\begin{aligned}
&\int_{T(I_*)} \operatorname{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z\right) \, dm(z) \\
&\qquad \le m(T(I_*))^{1/q} \left(\int_{T(I_*)} \operatorname{osc}\left(g \circ T|_{I_*}^{-1}, \varepsilon, z\right)^p \, dm(z) \right)^{1/p} \\
&\qquad \le m(T(I_*))^{1/q} \left(2\varepsilon \operatorname{Var}_p \left(g \circ T|_{I_*}^{-1}\right)^p \right)^{1/p} \\
&\qquad \le 2^{1/p} \Gamma_+^{1-1/p} \varepsilon^{1/p} \operatorname{Var}_p (g|_{I_*}) \\
&\qquad \le 2\delta_\dagger \varepsilon^{1/p} \Gamma_-^{1-1/p} \\
&\qquad = \frac{\delta \gamma^{1/p}}{18} A^{1-1/p} \varepsilon^{1/p}.
\end{aligned}$$

Together with (2.7), the definitions of Λ and K, and Lemma 2.7 with $\varepsilon = A$, this gives us

$$\begin{aligned}
&\frac{\operatorname{osc}_1(\mathscr{L}_g(h), \varepsilon)}{\varepsilon^{1/p}} \\
&\le 6 \frac{\operatorname{osc}_1(h, \gamma\varepsilon)}{\varepsilon^{1/p}} + \frac{4\varepsilon^{1-1/p}}{\Lambda} \|h\|_1 \\
&\quad + \delta \gamma^{1/p} A^{1-1/p} \sum_{\substack{I_* \in \mathscr{P} \\ m(I_*) > 0}} \left(\frac{1}{A} \int_{I_*} \operatorname{osc}(h|_{I_*}, A, y) \, dm(y) + \frac{1}{m(I_*)} \left| \int_{I_*} h \, dm \right| \right) \\
&\le 6\gamma^{1/p} \frac{\operatorname{osc}_1(h, \gamma\varepsilon)}{(\gamma\varepsilon)^{1/p}} + \delta\gamma^{1/p} \frac{\operatorname{osc}_1(h, A)}{A^{1/p}} + \frac{(4 + \delta\gamma^{1/p}) A^{1-1/p}}{\Lambda} \|h\|_1 \\
&\le (6 + \delta) \gamma^{1/p} \operatorname{Var}_{1/p,1}(h) + K \|h\|_1.
\end{aligned}$$

Since ε is an arbitrary in $(0, A]$, this proves the lemma. □

The following lemma is needed in the proof of Theorem 6.

Proof of Theorem 6. Put $\gamma := 8^{-p}$ and let $k \ge 1$ be an integer such that $\|g_k\|_\infty \le \gamma$. It is easy to check that g_k is also bounded p-variation. By Proposition 2.9 with g replaced by g_k, T replaced by T^k, and with $\delta = 1$, there are $A > 0$ and $K > 0$ such that for each function h in $\mathrm{H}^{1/p,1}(m)$, we have

$$\operatorname{Var}_{1/p,1}(\mathscr{L}_g^k(h)) \le \frac{7}{8} \operatorname{Var}_{1/p,1}(h) + K \|h\|_1.$$

Using that $\mathscr{L}_g$ is a positive linear contraction on $\mathrm{L}^1(m)$, we obtain

$$\begin{aligned}
\left\| \mathscr{L}_g^k(h) \right\|_{1/p,1} &= \operatorname{Var}_{1/p,1}(\mathscr{L}_g^k(h)) + \left\| \mathscr{L}_g^k(h) \right\|_1 \\
&\le \frac{7}{8} \operatorname{Var}_{1/p,1}(h) + (K+1) \|h\|_1.
\end{aligned}$$

This proves (2.3) $\beta = \frac{7}{8}$ and $C = K + 1$.

In view of parts 1 and 2 of Proposition 2.5, the assertions of 1 and 2 and are a direct consequence of the inequality (2.3) and the ergodic theorem of Ionescu-Tulcea and Marinescu [ITM50], see also [PU10, Theorem 5.5.5].

It remains to prove part 3. For each integer $n \geq 1$ put $h_n := \frac{1}{n}\sum_{j=0}^{n-1} \mathscr{L}_g^j(\mathbb{1})$. Since $\mathscr{L}_g$ is a positive operator, we have $h_n \geq 0$, and by part 4 of Proposition (2.5) the function h_n is bounded. So, by hypothesis H3 we have

$$\int h_n \, dm = \frac{1}{n}\sum_{j=0}^{n-1} \int \mathscr{L}_g^j(\mathbb{1}) \, dm = 1. \tag{2.8}$$

On the other hand, by part 2 for every integer $j \geq 1$ we have $\mathscr{L}_g^j = \mathscr{Q}^j + \sum_{\lambda \in \mathscr{E}} \lambda^j \mathscr{P}(\lambda)$. It follows that

$$\begin{aligned} \|h_n\|_{1/p,1} &= \left\| \frac{1}{n}\sum_{j=0}^{n-1} \left(\mathscr{Q}^j(\mathbb{1}) + \sum_{\lambda \in \mathscr{E}} \lambda^j \mathscr{P}(\lambda)(\mathbb{1}) \right) \right\|_{1/p,1} \\ &\leq \frac{1}{n}\sum_{j=0}^{n-1} \left\| \mathscr{Q}^j(\mathbb{1}) \right\|_{1/p,1} + \sum_{\lambda \in \mathscr{E}} \left(\frac{1}{n}\sum_{j=0}^{n-1} \lambda^j \right) \left\| \mathscr{P}(\lambda)(\mathbb{1}) \right\|_{1/p,1}. \end{aligned}$$

If assume that 1 is not an eigenvalue of $\mathscr{L}_g$, then we obtain $\|h_n\|_{1/p,1} \to 0$ as $n \to +\infty$. But this contradicts (2.8), so 1 must be an eigenvalue of $\mathscr{L}_g$, and the argument above also proves that $h_n \to h$ as $n \to +\infty$ in $\mathrm{H}^{1/p,1}(m)$. On the other hand,

$$\mathscr{L}_g(h) = \mathscr{L}_g(\mathscr{P}(1)(\mathbb{1})) = \mathscr{P}(1)^2(\mathbb{1}) = \mathscr{P}(1)(\mathbb{1}) = h.$$

So, for every continuous function $\phi : X \to \mathbb{C}$ we have

$$\begin{aligned} \int \phi \circ T \, d\nu = \int (\phi \circ T) h \, dm = \int \mathscr{L}_g((\phi \circ T)h) \, dm = \int \phi \mathscr{L}_g(h) \, dm \\ = \int \phi h \, dm = \int \phi \, d\nu. \end{aligned}$$

This proves that the measure ν is invariant by T, and completes the proof of the theorem. □

2.5. **Topologically exact maps.** The purpose of this section is to prove Corollary 2.4. We use the notation introduced in §2.2.1.

Lemma 2.12. *Given α in $(0, 1]$, let C_* be given by part 4 of Proposition 2.5. Then for every h and g in $\mathrm{H}^{\alpha,1}(m)$, we have*

$$\|h \cdot g\|_{\alpha,1} \leq 2C_* \|h\|_{\alpha,1} \cdot \|g\|_{\alpha,1}.$$

Proof. Using that each of the functions h and g is represented by a bounded function, we have

$$\operatorname{osc}(h \cdot g, \varepsilon, x) \leq \|h\|_\infty \operatorname{osc}(g, \varepsilon, x) + \|g\|_\infty \operatorname{osc}(h, \varepsilon, x).$$

We thus have

$$|h \cdot g|_{\alpha,1} \le \|h\|_\infty \cdot |g|_{\alpha,1} + \|g\|_\infty \cdot |h|_{\alpha,1},$$

and using part 4 of Proposition 2.5 twice, we have

$$\begin{aligned}\|h \cdot g\|_{\alpha,1} &\le \|h\|_\infty \cdot \|g\|_1 + \|h\|_\infty \cdot |g|_{\alpha,1} + \|g\|_\infty \cdot |h|_{\alpha,1} \\ &\le \|h\|_\infty \cdot \|g\|_{\alpha,1} + C_* |h|_{\alpha,1} \|g\|_{\alpha,1} \\ &\le 2C_* \|h\|_{\alpha,1} \cdot \|g\|_{\alpha,1}.\end{aligned}$$

□

Proof of Corollary 2.4. By part 3 of Theorem 6 the function $h := \mathscr{P}(1)(\mathbf{1})$ satisfies $\int h \, dm = 1$ and the measure $\nu := hm$ is a probability. On the other hand, each of the spaces

$$\mathrm{E}_0 := \left\{ \psi \in \mathrm{H}^{1/p,1}(m) : \int \psi \, dm = 0 \right\} \text{ and } \mathrm{E}_1 := \{\alpha h : \alpha \in \mathbb{C}\}$$

is invariant by $\mathscr{L}_g$. Moreover, since each function ψ in $\mathrm{H}^{1/p,1}(m)$ can be decomposed as

$$\psi = \left(\int \psi \, dm \right) h + \left(\psi - \left(\int \psi \, dm \right) h \right),$$

we have $\mathrm{H}^{1/p,1}(m) = E_1 \oplus E_0$.

1. Let λ be an eigenvalue of $\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)}$ satisfying $|\lambda| = 1$, and let ϕ be a nonzero element of $\mathrm{H}^{1/p,1}(m)$ such that $\mathscr{L}_g(\phi) = \lambda\phi$. By hypothesis H3, for each integer $n \ge 1$ we have

$$\int |\phi| \, dm = \int |\lambda^n \phi| \, dm = \int |\mathscr{L}_g^n(\phi)| \, dm \le \int \mathscr{L}_g^n(|\phi|) \, dm = \int |\phi| \, dm,$$

and therefore

$$\int \mathscr{L}_g^n(|\phi|) - |\mathscr{L}_g^n(\phi)| \, dm = 0.$$

Since $\mathscr{L}_g^n(|\phi|) - |\mathscr{L}_g^n(\phi)| \ge 0$, it follows that we have

$$\mathscr{L}_g^n(|\phi|) = |\mathscr{L}_g^n(\phi)| = |\phi| \tag{2.9}$$

on a set of full measure with respect to m.

In part 1.1 below we prove that ϕ is nonzero on a set of full measure with respect to m, and in part 1.2 we show that there is θ_0 in $\mathbb{R}$ such that $\phi = \exp(i\theta_0)|\phi|$ in $\mathrm{H}^{\alpha,1}(m)$. Using these facts, we complete the proof of part 1 of the corollary in part 1.3.

1.1. Since ϕ is nonzero, there is $\kappa_0 > 0$ such that $\{x \in X : |\phi| \ge \kappa_0\}$ has positive measure with respect to m. Let Y be the set of density points of this set, so $m(Y) > 0$, and put

$$\varepsilon_0 := \min \left\{ \left(\frac{m(Y)\kappa_0}{2\||\phi|\|_{\alpha,1}} \right)^{\frac{1}{\alpha}}, A \right\}.$$

Note that for each y in Y, the number

$$\kappa(y) := \text{ess-inf}\{|\phi(x)| : x \in B_d(y, \varepsilon_0)\}$$

satisfies

$$\text{osc}(|\phi|, \varepsilon_0, y) \geq \kappa_0 - \kappa(y),$$

so

$$\int_Y \kappa_0 - \kappa(y)\ dm(y) \leq \text{osc}_1(|\phi|, \varepsilon_0) \leq \varepsilon_0^\alpha \||\phi|\|_{\alpha,1} \leq m(Y)\kappa_0/2.$$

This implies that there is y_0 in Y such that

$$\text{ess-inf}\{|\phi(x)| : x \in B_d(y_0, \varepsilon_0)\} = \kappa(y_0) \geq \kappa_0/2. \tag{2.10}$$

Since by hypothesis f is topologically exact on X, there is an integer $n \geq 1$ such that $T^n(B_d(y_0, \varepsilon_0)) = X$. Combined with hypothesis H2 and (2.9), the estimate (2.10) implies that $|\phi|$ is nonzero on a set of full measure with respect to m.

1.2. By part 1.1 there is $\upsilon > 0$ such that

$$W := \{x \in X : |\phi(x)| \geq \upsilon\}$$

satisfies $m(W) \geq 1/2$. Let W' be the set of density points of W, so $m(W') \geq 1/2$.

For each x in X such that $\phi(x) \neq 0$, let $\theta(x)$ in $\mathbb{R}/2\pi\mathbb{Z}$ be such that $\phi(x) = \exp(i\theta(x))|\phi(x)|$. Suppose by contradiction that the function $\theta : X \to \mathbb{R}$ so defined is not constant on a set of full measure with respect to m. Then there are disjoint closed intervals Θ and Θ' such that the sets $\theta^{-1}(\Theta)$ and $\theta^{-1}(\Theta')$ are disjoint, and such that each of these sets has positive measure with respect to m. By property (2.9), for every integer $n \geq 1$ we have

$$T^{-n}\left(T^n\left(\theta^{-1}(\Theta)\right)\right) \setminus \{|\phi| = 0\} \subset \theta^{-1}(\Theta).$$

Combined with part 1, hypothesis H2, and with our hypothesis that T is topologically exact on X, this implies that for every y and every $\varepsilon > 0$ the set $\theta^{-1}(\Theta)$ intersects $B_d(y, \varepsilon)$ on a set of positive measure. The same property holds replacing Θ by Θ'. Thus, if we denote by δ the distance between Θ and Θ' in $\mathbb{R}/2\pi\mathbb{Z}$, then for every y in W' and every ε in $(0, A]$, we have

$$\text{osc}(\phi, \varepsilon, y) \geq 2\upsilon \sin(\delta/2).$$

Therefore

$$\|\phi\|_{\alpha,1} \geq \frac{\text{osc}_1(\phi, \varepsilon)}{\varepsilon^\alpha} \geq \frac{m(W')(2\upsilon\sin(\delta/2))}{\varepsilon^\alpha} \geq \frac{\upsilon\sin(\delta/2)}{\varepsilon^\alpha}.$$

Since this holds for an arbitrary ε in $(0, A]$, we obtain a contradiction. This contradiction shows that the function θ is constant on a set of full measure with respect to m.

1.3. By part 1.2 there is θ_0 in $\mathbb{R}$ such that $\phi = \exp(i\theta_0)|\phi|$ in $\mathrm{H}^{\alpha,1}(m)$. It follows that $\mathscr{L}_g(|\phi|) = \lambda|\phi|$ is nonnegative, and therefore that $\lambda = 1$. Since by part 3 of Theorem 6 the number 1 is an eigenvalue of $\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)}$, this proves that the number 1 is the only eigenvalue of $\mathscr{L}_g$ of modulus 1.

The existence of ρ in $(0,1)$ such that the spectrum of $\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)}$ is contained in $B(0,\rho)\cup\{1\}$ follows from part 2 of Theorem 6.

It remains to prove that the algebraic multiplicity of 1 as an eigenvalue of $\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)}$ is 1. Denote by Id the identity operator of $\mathrm{H}^{1/p,1}(m)$, and let ϕ be in the kernel of $\left(\mathscr{L}_g|_{\mathrm{H}^{1/p,1}(m)} - \mathrm{Id}\right)^2$. Then $\widetilde{\phi} := \mathscr{L}_g(\phi) - \phi$ satisfies $\mathscr{L}_g(\widetilde{\phi}) = \widetilde{\phi}$. Suppose $\widetilde{\phi}$ is nonzero. Then we can apply parts 1.1 and 1.2 with ϕ replaced by $\widetilde{\phi}$, to conclude that there is $\widetilde{\theta}_0$ in $\mathbb{R}$ such that $\widetilde{\phi} = \exp\left(i\widetilde{\theta}_0\right)|\widetilde{\phi}|$ in $\mathrm{H}^{\alpha,1}(m)$. Using hypothesis H3 we obtain

$$\begin{aligned} 0 &< \int |\widetilde{\phi}| \; dm \\ &= \exp\left(-i\widetilde{\theta}_0\right) \int \mathscr{L}_g(\phi) - \phi \; dm \\ &= \exp\left(-i\widetilde{\theta}_0\right) \left(\int \mathscr{L}_g(\phi) \; dm - \int \phi \; dm \right) \\ &= 0. \end{aligned}$$

This contradiction proves that $\mathscr{L}_g(\phi) - \phi = \widetilde{\phi}$ is zero, and completes the proof of part 1.

2. Let C_* be the constant given by part 4 of Proposition 2.5 and let ρ and M be the constants given by part 2 of Theorem 6. Putting $\widehat{\psi} = \psi - \int_X \psi \; d\nu$, we have

$$\begin{aligned} C_n(\phi,\psi) &= \left| \int \phi \circ f^n \cdot \widehat{\psi} \; d\nu \right| \\ &= \left| \int (\phi \circ f^n) \cdot \widehat{\psi} \cdot h \; dm \right| \\ &= \left| \int \mathscr{L}_g^n \left((\phi \circ f^n) \cdot \widehat{\psi} \cdot h \right) \; dm \right| \\ &= \left| \int \phi \cdot \mathscr{L}_g^n \left(\widehat{\psi} \cdot h \right) \; dm \right| \\ &\le \|\phi\|_\infty \cdot \left\| \mathscr{L}_g^n \left(\widehat{\psi} \cdot h \right) \right\|_1 \\ &\le \|\phi\|_\infty \cdot \left\| \mathscr{L}_g^n \left(\widehat{\psi} \cdot h \right) \right\|_{1/p,1}. \end{aligned}$$

Noting that $\widehat{\psi} \cdot h$ is in E_0 and using part 2 of Theorem 6, we conclude that

$$C_n(\phi,\psi) \le M \|\phi\|_\infty \|\widehat{\psi} \cdot h\|_{1/p,1} \rho^n. \tag{2.11}$$

On the other hand, by Lemma 2.12 we have

$$\begin{aligned}\|\widehat{\psi}\cdot h\|_{1/p,1} &\le 2C_*\|\widehat{\psi}\|_{1/p,1}\cdot\|h\|_{1/p,1}\\ &\le \left(2C_*\|h\|_{1/p,1}\right)\left(\|\psi\|_{1/p,1}+\|\psi\|_1\cdot\|h\|_\infty\right)\\ &\le \left(2C_*\|h\|_{1/p,1}\left(1+\|h\|_\infty\right)\right)\|\psi\|_{1/p,1}.\end{aligned}$$

Together with (2.11) this implies the desired inequality with $C = 2MC_*\|h\|_{1/p,1}\left(1+\|h\|_\infty\right)$.

3. Let C_* be the constant given by part 4 of Proposition 2.5. Observe that for each τ in $\mathbb{C}$, we have

$$|\exp(\tau\psi)|_{1/p,1} \le \exp(|\tau|\cdot\|\psi\|_\infty)|\tau|\cdot|\psi|_{1/p,1},$$

so the function $\exp(\tau\psi)$ is in $\mathrm{H}^{1/p,1}(m)$. Thus, by Lemma 2.12 for every χ in $\mathrm{H}^{1/p,1}(m)$ we have

$$\|\mathscr{L}_\tau(\chi)\|_{1/p,1} \le \left(2C_*\|\mathscr{L}_g\|_{1/p,1}\cdot\|\exp(\tau\psi)\|_{1/p,1}\right)\|\chi\|_{1/p,1}.$$

This proves that $\mathscr{L}_\tau$ maps $\mathrm{H}^{1/p,1}(m)$ to itself and that $\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is bounded.

To prove that $\tau\mapsto\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is analytic in the sense of Kato, for each ε in $\mathbb{C}$ let $\eta_\varepsilon:\mathbb{C}\to\mathbb{C}$ be defined by $\eta_\varepsilon(z) := \frac{\exp(\varepsilon z)-1}{\varepsilon} - z$ and put $\psi_\varepsilon := \eta_\varepsilon\circ\psi$. Noting that $D\eta_\varepsilon(z) = \exp(\varepsilon z)-1$, we have

$$|\psi_\varepsilon| \le \left(\exp(|\varepsilon|\cdot\|\psi\|_\infty)-1\right)|\psi|$$

on X, and

$$|\psi_\varepsilon|_{1/p,1} \le \left(\exp(|\varepsilon|\cdot\|\psi\|_\infty)-1\right)|\psi|_{1/p,1}.$$

It follows that

$$\|\psi_\varepsilon\|_{1/p,1} \le \left(\exp(|\varepsilon|\cdot\|\psi\|_\infty)-1\right)\|\psi\|_{1/p,1}. \tag{2.12}$$

On the other hand, if for each τ in $\mathbb{C}$ we define the operator $\mathscr{D}_\tau$ by $\mathscr{D}_\tau(\chi) := \mathscr{L}_\tau(\psi\cdot\chi)$, then for every ε in $\mathbb{C}$ and every χ in $\mathrm{H}^{1/p,1}(m)$ we have

$$\frac{\mathscr{L}_{\tau+\varepsilon}(\chi)-\mathscr{L}_\tau(\chi)}{\varepsilon} - \mathscr{D}_\tau(\chi) = \mathscr{L}_\tau(\psi_\varepsilon\cdot\chi).$$

Combined with (2.12), we have by Lemma 2.12

$$\begin{aligned}&\left\|\frac{\mathscr{L}_{\tau+\varepsilon}(\chi)-\mathscr{L}_\tau(\chi)}{\varepsilon} - \mathscr{D}_\tau(\chi)\right\|_{1/p,1}\\ &\qquad\le 2C_*\|\mathscr{L}_\tau\|_{1/p,1}\|\psi_\varepsilon\|_{1/p,1}\|\chi\|_{1/p,1}\\ &\qquad\le \left(2C_*\|\mathscr{L}_\tau\|_{1/p,1}\|\psi\|_{1/p,1}\right)\cdot\left(\exp(|\varepsilon|\cdot\|\psi\|_\infty)-1\right)\|\chi\|_{1/p,1}.\end{aligned}$$

This implies that the operator norm $\left\|\frac{\mathscr{L}_{\tau+\varepsilon}-\mathscr{L}_\tau}{\varepsilon} - \mathscr{D}_\tau\right\|_{1/p,1}$ converges to 0 as ε converges to 0, and completes the proof that $\tau\mapsto\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is analytic in the sense of Kato.

That the spectral radius of $\mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ depends on a real analytic way on τ on a neighborhood of $\tau = 0$ follows from part 1 and from the fact that $\tau \mapsto \mathscr{L}_\tau|_{\mathrm{H}^{1/p,1}(m)}$ is analytic in the sense of Kato, see for example [RS78, Theorem XII.8]. This completes the proof of the corollary. □

2.6. **Exercises.**

Exercise 2.1. Let I be an interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuously differentiable function whose derivative vanishes at most on a finite set. Show that, if we denote by Leb the Lebesgue measure on $\mathbb{R}$, then the measure $f_* \operatorname{Leb}|_I$ is absolutely continuous with respect to Leb, and that for x in a set of full Lebesgue measure in I we have

$$\frac{df_* \operatorname{Leb}|_I}{d \operatorname{Leb}}(x) = \sum_{y \in f^{-1}(x)} |Df(y)|^{-1}.$$

Exercise 2.2. Let I be a compact interval of $\mathbb{R}$ and let $f : I \to I$ be a piecewise expanding map.

A. Under what circumstances the function

$$x \mapsto \sum_{y \in f^{-1}(x)} |Df(y)|^{-1}$$

is continuous?

B. Under what circumstances we have for each continuous function $\psi : I \to \mathbb{R}$ that the function $\mathscr{L}(\psi) : I \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathscr{L}(\psi)(x) := \sum_{y \in f^{-1}(x)} |Df(y)|^{-1},$$

continuous?

Exercise 2.3. Let T be a map as in §2.2.2, and assume in addition that T is differentiable with Hölder continuous derivative on each element of the partition $\mathscr{P}$. Moreover, assume that for some constant $\lambda > 1$ and some integer $N \geq 1$ we have $|DT^N| \geq \lambda$. Show that if we choose $g = 1/|DT|$ and $m = \operatorname{Leb}|_I$, then the hypotheses H1, H2, and H3 of §2.2.2 are satisfied.

Exercise 2.4. Let T, g, $\mathscr{L}_g$, and m be as in §2.2.2. Show that $\mathscr{L}_g$ maps the space BV_p into itself. Does a 2 norm inequality, like Proposition 2.9, hold on BV_p?

Exercise 2.5. Fix $p \geq 1$, a compact subset X of $\mathbb{R}$, and let $g : X \to \mathbb{R}$ be a function of bounded p-variation. Prove the following facts used in the proof of Theorem 6.

A. For every $\delta > 0$ there is a finite partition $\mathscr{P}$ of X into intervals (for the corresponding order relation), such that for each P in $\mathscr{P}$ we have

$$\operatorname{Var}_p(g|_P) \leq \delta.$$

B. If $T : I \to I$ is a piecewise monotone map, then for every integer $n \geq 2$ the function $g_n : I \to \mathbb{R}$ defined by

$$g_n := g \cdot g \circ T \cdot \cdots \cdot g \circ T^{n-1}$$

is of bounded p-variation.

Exercise 2.6. Given m, α, and $\mathrm{H}^{\alpha,1}(m)$ as in §2.2.1, prove the following statements that are some of the hypotheses of the ergodic theorem of Ionescu-Tulcea and Marinescu [ITM50].

A. Let $K > 0$ and let $(h_n)_{n=1}^{+\infty}$ be a sequence of functions in $\mathrm{H}^{1,\alpha}(m)$ such that for every n we have $\|h_n\|_{\alpha,1} \leq K$. If $(h_n)_{n=1}^{+\infty}$ converges in $L^1(m)$ to a function h, then h is in $\mathrm{H}^{\alpha,1}(m)$ and $\|h\|_{\alpha,1} \leq K$.
B. Given a bounded subset B of $\mathrm{H}^{1,\alpha}(m)$, prove that the closure of $\mathscr{L}_g(B)$ in $L^1(m)$ is compact.

3. Smooth maps

The purpose of this section is to give some elements in the proof of the following result, which is one of the weakest criterion's available for the existence of an acip for smooth interval maps.

Throughout this section we fix a compact interval I of $\mathbb{R}$. A smooth interval map $f : I \to I$ is *non-degenerate* if the set of points of I at which the derivative of f vanishes is finite, and if at each of these points there is a higher order derivative of f that is non-zero. For a non-degenerate smooth map $f : I \to I$, a periodic point p of period $n \geq 1$ is *hyperbolic repelling* if $|Df^n(p)| > 1$.

Theorem 7 ([BRLSvS08], Main Theorem)**.** *Let $f : I \to I$ be a non-degenerate smooth map having all periodic points hyperbolic repelling. Assume that for each critical point c of f, we have*

$$\lim_{n\to+\infty} |Df^n(f(c))| = +\infty. \tag{3.1}$$

Then f has an acip with respect to the Lebesgue measure.

See also [RLS14, Theorem I] for an alternative proof, and [BSvS06] for the case of unimodal maps. Previous results on the existence of acip's were obtained in [BLVS03, CE80, KN92, Mis81, NvS91, You92], among others.

The smoothness assumption can be relaxed substantially; the assumption that at each critical point a high order derivative is non-zero is then replaced by a "non-flatness" condition. However, as in many results of this type, the finiteness of critical points and the non-flatness assumption are both used in an essential way.

The Main Theorem of [RLS14] gives a quantitative version of Theorem 7: For each $\ell > 1$ there is a constant $K > 0$ such that, if f is a non-degenerate smooth map all whose critical points have order at most ℓ, and such that for each critical point c of f we have

$$\liminf_{n\to+\infty} |Df^n(f(c))| \geq K,$$

then f has an acip. See also [BRLSvS08, Main Theorem'] for a version in which K also depends on the number of critical points of f. There are examples showing that the condition $\liminf_{n\to+\infty} |Df^n(f(c))| > 0$ is neither sufficient, or necessary for the existence of an acip, see [BKNvS96] and [Bru94], respectively.

3.1. On the regularity of the acip.

For a non-degenerate smooth map $f : I \to I$, denote by $\mathrm{Crit}(f)$ the set of critical points of f. The *order* of a critical point c of f is the least integer $\ell_c \geq 2$ such that the ℓ_c-th derivative $f^{(\ell_c)}(c)$ is non-zero. We also put

$$\ell_{\max}(f) := \max\{\ell_c : c \in \mathrm{Crit}(f)\}.$$

For a measurable subset A of I denote by $|A|$ its Lebesgue measure.

The proof of Theorem 7 given in [BRLSvS08] is based on the following estimate: For every κ in $(0,1)$ there exists $M>0$ so that for each Borel set A, we have

$$|f^{-n}(A)| \leq M|f(A)|^{\frac{\kappa}{\ell_{\max}}}. \tag{3.2}$$

The estimate for a general Borel set A is obtained from the special case in which A is an interval through the "sliding argument", used in an earlier result of Nowicki and van Strien in [NvS91]. One of the features of this method is that it gives an estimate on the regularity of the density of the acip: For each p in $[1, \ell_{\max}(f)/(\ell_{\max}(f)-1))$, the density of the acip is in the space $L^p(\mathrm{Leb})$, see below for a proof of this fact. This estimate is optimal: The results of Ledrappier in [Led81] imply that for $p = \ell_{\max}(f)/(\ell_{\max}(f)-1)$, the density of the acip does not belong to the space $L^p(\mathrm{Leb})$.

Recently, L^p estimates on the density of the acip have been used in [AFLV11] to understand mixing rates of physical measures in arbitrary dimension. So the following problem arises naturally.

Problem 3.1. *Let f be a non-degenerate smooth interval map having an acip ν with respect to the Lebesgue measure. Does there exist $p>1$ such that the density of ν is in the space $L^p(\mathrm{Leb})$?*

Under the additional assumption that ν is exponentially mixing, this problem is solved affirmatively [RL12b, Corollary B].

Unfortunately, the sliding argument used in [BRLSvS08, NvS91] only seems to work for interval maps and the Lebesgue measure as a reference measure. The proof of Theorem 7 given in [RLS14], based on an induced scheme, applies simultaneously to real and complex maps, and also gives an estimate on mixing rates.

We now explain how to obtain the L^p estimate of the density of the acip form (3.2). We use the argument given above Theorem A in [NvS91], see also [dMvS93, p. 378]. Let N be a positive integer and $p \in [1, \frac{\ell_{\max}}{\ell_{\max}-1})$ be given. Then choose $\kappa \in (0,1)$ sufficiently close to 1 so that $1-\frac{\kappa}{\ell_{\max}} > p$, let $C>0$ be such that for every measurable set A we have $|f(A)| \leq C|A|$, and put $M' = MC^{\frac{\kappa}{\ell_{\max}}}$. It follows from the inequality (3.2) that, if we denote the Lebesgue measure by m, then for each $n \geq 1$ the measure

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}(f^i)_* m$$

is such that for every measurable set A we have $\mu_n(A) \leq M'|A|^{\frac{\kappa}{\ell_{\max}}}$. Fix an accumulation point μ in the weak* topology of the sequence $(\mu_n)_{n\geq 1}$. Then μ is f-invariant and for every measurable set A we have $\mu(A) \leq M'|A|^{\frac{\kappa}{\ell_{\max}}}$. In particular μ is absolutely continuous with respect to the Lebesgue measure; we denote by ρ its density. For each $k \geq 0$ put $D_k = \{\rho^p \geq k\}$ and observe that

$$\int \rho^p dx \leq \sum_{k \geq 0} (k+1)|D_k \setminus D_{k+1}| = \sum_{k \geq 0} |D_k|.$$

For each $k \geq 1$ we have

$$k^p |D_k| \leq \mu(D_k) \leq M' |D_k|^{\frac{\kappa}{\ell_{\max}}},$$

and $|D_k| \leq k^{p(1-\frac{\kappa}{\ell_{\max}})^{-1}}$. By the choice of κ we have $p(1 - \frac{\kappa}{\ell_{\max}})^{-1} > 1$, so $\sum_{k \geq 0} |D_k| < \infty$. This shows that $\rho \in L^p$.

3.2. **Backward contraction and nice sets.** In the proofs of Theorem 7 in [BRLSvS08] and in [RLS14], a weak hyperbolicity condition is used as an intermediate step. This weak hyperbolicity condition is known as "backward contraction", and it is defined below. In fact, the hypothesis (3.1) in Theorem 7 is only used to prove that the map f is backward contracting, see [BRLSvS08, Theorem 1]. Then it is shown that every backward contracting map having all periodic points hyperbolic repelling has an acip, see [BRLSvS08, §2] and [RLS14, Main Theorem].

To define the backward contraction condition, let $f : I \to I$ be a non-degenerate smooth map, and let $\mathrm{CV}(f) := f(\mathrm{Crit}(f))$ be the set of critical values of f. For each c in $\mathrm{Crit}(f)$ and each $\delta > 0$, denote by $\widetilde{B}(c, \delta)$ the connected component of $f^{-1}(f(c))$ containing c. Note that for δ small, the set $\widetilde{B}(c, \delta)$ is an interval whose length is comparable to δ^{1/ℓ_c}, up to a multiplicative constant independent of δ.

Definition 3.2. Let f be a non-degenerate smooth map having all periodic points hyperbolic repelling. Then f is *backward contracting*, if for every $r > 1$ and every sufficiently small δ the following property holds: For each c in $\mathrm{Crit}(f)$, each integer $n \geq 1$, and each connected component W of $f^{-n}(\widetilde{B}(c, r\delta))$,

$$\mathrm{dist}(W, \mathrm{CV}(f)) < \delta \text{ implies } |W| < \delta. \tag{3.3}$$

The backward contraction condition was first introduced in [RL07], in the complex setting. Li and Shen showed that for a non-degenerate smooth map having all periodic points hyperbolic repelling, the backward contraction condition is actually equivalent to (3.1), see [LS10, Theorems A' and B']. See also [LS13], [RL07, Appendix A], and [RLS13, Theorem 1] for related results.

One of the features of backward contracting maps that is used in a crucial way in [BRLSvS08] and in [RLS14], is the existence of "nice sets" at every scale. We proceed to state this more precisely. For a non-degenerate smooth map $f : I \to I$ having all periodic points hyperbolic repelling, an open subset V of I is a *nice set for f*, if the following hold:

- Each connected component of V contains precisely one critical point of f;
- For every integer $n \geq 1$, the set $f^n(\partial V)$ is disjoint from V.

For a nice set V and a critical point c of f, we denote by V^c the connected component of V containing c.

For a non-degenerate smooth map that is topologically exact, arbitrarily small nice sets can be easily constructed using periodic points. Roughly speaking, the following property asserts that for a backward contracting map there is a nice set at every scale.

Proposition 3.3 ([BRLSvS08], Proposition 3)**.** *Let $f : I \to I$ be a non-degenerate smooth map that is backward contracting. Then for every sufficiently small $\delta > 0$ and each critical point c of f, there is an open interval V^c satisfying*

$$\widetilde{B}(c, \delta) \subset V^c \subset \widetilde{B}(c, 2\delta),$$

and such that $\bigcup_{c \in \mathrm{Crit}(f)} V^c$ is a nice set for f.

The (short) proof of this result is below. It follows the proof of the analogous statement in the complex setting [RL07, Lemma 6.2].

Proof. Let $\delta > 0$ be sufficiently small so that the backward contracting property holds with $r = 2$. For each integer $n \geq 0$, put

$$V_n := \bigcup_{j=0}^{n} \bigcup_{c \in \mathrm{Crit}(f)} f^{-j}(\widetilde{B}(c, \delta)),$$

and for each critical point c of f denote by V_n^c the connected component of V_n containing c. Note that $V_\infty := \bigcup_{n=1}^{+\infty} V_n$ is such that for every integer $j \geq 1$ the set $f^j(\partial V_\infty)$ is disjoint from V_∞. Thus, it is enough to show that for every critical point c of f, the connected component V_∞^c of V_∞ containing c is contained in $\widetilde{B}(c, 2\delta)$. Since we clearly have $V_\infty^c = \bigcup_{n=1}^{+\infty} V_n^c$, it is enough to show that for each integer $n \geq 1$ the set V_n^c is contained in $\widetilde{B}(c, 2\delta)$. We proceed by induction in n, the case $n = 0$ being trivial. Let $n \geq 1$ be an integer such that for each critical point c of f, the set V_{n-1}^c is contained in $\widetilde{B}(c, 2\delta)$. Let Z be a connected component of $f(V_n^c) \setminus B(f(c), \delta)$. Note that for each z in Z there is an integer $m(z)$ in $\{0, \ldots, n-1\}$ and a critical point $c(z)$ of f, such that $f^{m(z)}(z)$ is in $\widetilde{B}(c(z), \delta)$. Let z_0 be a point in Z for which $m_0 := m(z_0)$ is minimal, and put $c_0 := c(z_0)$. Then $f^{m_0}(Z)$ is contained in $V_{n-m_0-1}^{c_0} \subset V_{n-1}^{c_0}$. By the induction hypothesis we conclude that $f^{m_0}(Z)$ is contained in $\widetilde{B}(c_0, 2\delta)$. Then the backward contracting property implies that $|Z| < \delta$. This proves that $f(V_n^c)$ is contained in $B(f(c), 2\delta)$, and therefore that V_n^c is contained in $\widetilde{B}(c, 2\delta)$. This completes the proof of the proposition. □

3.3. **Proving backward contraction.** In this section we give a proof of the following result, which is an intermediate step in the proof of Theorem 7. Given $N \geq 1$, $\ell_{\max} > 1$ and $K > 0$, denote by $\mathcal{A}(N, \ell_{\max}, K)$ the collection of all non-degenerate smooth maps $f : I \to I$ that have at most N critical points, such that every critical point has order at most $\ell_{\max}$ and such that for every critical value v we have

$$\liminf_{n\to+\infty} |Df^n(v)| \geq K.$$

Theorem 8. *For real numbers $\ell_{\max} > 1$ and $r > 1$, there exists $K = K(r, \ell_{\max})$ such that if f is a map in the class $\mathcal{A}(N, \ell_{\max}, K)$ for some $N \geq 1$, then it satisfies property $BC(r)$.*

A sequence of open intervals $\{G_j\}_{j=0}^{s}$ is called *a chain* if for each $0 \leq j < s$, G_j is a component of $f^{-1}(G_{j+1})$. The *order* of the chain is defined to be the number of j's with $0 \leq j < s$ and such that G_j contains a critical point.

For each critical point c and $\varepsilon > 0$, let $\widetilde{B}(c, \varepsilon)$ be the connected component of $f^{-1}((f(c) - \varepsilon, f(c) + \varepsilon))$ containing c. Moreover, let

$$\widetilde{B}(\varepsilon) = \bigcup_{c \in \mathrm{Crit}(f)} \widetilde{B}(c, \varepsilon).$$

Note that provided that ε is small enough,

$$\widetilde{B}(c, r\varepsilon) \approx r^{1/\ell_c} \widetilde{B}(c, \varepsilon).$$

Lemma 3.4. *For any $\rho > 0$ and $\ell_{\max} > 1$, there exists $K > 1$, and for each $f \in \bigcup_{N=1}^{\infty} \mathcal{A}(N, \ell_{\max}, K)$ there exists $\varepsilon_0 > 0$ with the following property. Let $c, c' \in \mathrm{Crit}(f)$ and $\varepsilon \in (0, \varepsilon_0)$. If $f^s(c) \in \widetilde{B}(c', \varepsilon)$ for some $s \geq 1$, and if J is the component of $f^{-s}(\widetilde{B}(c', \varepsilon))$ containing c, then*

$$J \subset \widetilde{B}(c, \rho\varepsilon).$$

Proof. We may assume that $\rho \in (0, 1)$. Put $r = 2^{\ell_{c'}}$ and consider the chains $\{G_j\}_{j=0}^{s}$ and $\{H_j\}_{j=0}^{s}$ with $G_s = \widetilde{B}(c', r\varepsilon) \supset H_s = \widetilde{B}(c', \varepsilon)$ and $G_0 \supset H_0 = J$. Let $s_1 < s$ be maximal such that G_{s_1} contains a critical point c_1. Let H'_{s_1+1} be the convex hull of $H_{s_1+1} \cup \{f(c_1)\}$, and observe that $H'_{s_1+1} \subset G_{s_1+1}$.

Claim. Provided that ε is small enough and that K is large enough, we have

$$H_{s_1} \subset \widetilde{B}(c_1, \rho\varepsilon). \tag{3.4}$$

In fact, since

$$f^{s-s_1-1} : G_{s_1+1} \to G_s$$

is a diffeomorphism with $|G_s|$ small, it follows from the one-sided Koebe Principle (Proposition A.1 (ii)), applied to each of the connected components of $G_{s_1+1} \setminus \{f(c_1)\}$ intersecting H'_{s_1+1}, that for each $x \in H'_{s_1+1}$, we have

$$|Df^{s-s_1-1}(x)| \geq C|Df^{s-s_1-1}(f(c_1))|, \tag{3.5}$$

where $C > 0$ is a universal constant. Provided that ε is small enough, we have

$$|Df^{s-s_1}(f(c_1))| \geq K$$

by the hypothesis. Moreover, by non-flatness of the critical points there is a constant $C_1 > 0$ such that

$$|Df(f^{s-s_1}(c_1))| \le C_1 \frac{|fG_s|}{|G_s|}.$$

Thus

$$|Df^{s-s_1-1}(f(c_1))| = \frac{|Df^{s-s_1}(f(c_1))|}{|Df(f^{s-s_1}(c_1))|} \ge KC_1^{-1}\frac{|G_s|}{|fG_s|}.$$

This, equation (3.5) and the mean value theorem imply

$$\frac{|G_s|}{|H'_{s_1+1}|} \ge CC_1^{-1}K\frac{|G_s|}{|fG_s|},$$

which implies that

$$|H'_{s_1+1}| \le \rho\varepsilon$$

provided that K is sufficiently large. The claim follows.

If $s_1 = 0$ then the proof of the lemma is completed. For the general case, the lemma follows by an easy induction on s. □

Let us say that f satisfies *property $BC^*(r)$* if the following holds: for any $\varepsilon > 0$ small enough, $c, c' \in \text{Crit}(f)$, and $s \ge 1$, if $f^s(c) \in \widetilde{B}(c', r\varepsilon)$ and J is the component of $f^{-s}(\widetilde{B}(c', r\varepsilon))$ which contains c, then $J \subset \widetilde{B}(c, \varepsilon)$. (So the difference with property $BC(r)$ is that in equation (3.3), the assumption $\text{dist}(W, CV) < \varepsilon$ is replaced by $W \cap CV \neq \emptyset$.)

The above lemma can be reformulated as

Proposition 3.5. *For any $\ell_{\max} > 1$ and $r > 1$ there exists $K \ge 1$ such that each $f \in \bigcup_{N=1}^{\infty} \mathcal{A}(N, \ell_{\max}, K)$ satisfies property $BC^*(r)$.*

Property $BC^*(r)$ is closely related to $BC(r)$. Clearly the latter implies the former. The other direction is shown in the following proposition.

Lemma 3.6. *For any $f \in \mathcal{A}$, $BC^*(8^{\ell_{\max}} r)$ implies $BC(r)$, where $\ell_{\max}$ is the maximum of the order of the critical points of f.*

Proof. Let $\varepsilon > 0$ be a small constant. Then for all $c \in \text{Crit}(f)$, $\widetilde{B}(c, 8^{\ell_{\max}} r\varepsilon)$ contains the 3-scaled neighborhood of $\widetilde{B}(c, r\varepsilon)$.

Let $c, c' \in \text{Crit}(f)$ and $x \in \widetilde{B}(c, \varepsilon)$. Let $s \ge 1$ be such that $f^s(x) \in \widetilde{B}(c', r\varepsilon)$ and let J_k be the component of $f^{-(s-k)}(\widetilde{B}(c, r\varepsilon))$ which contains $f^k(x)$. We want to show that $|J_1| < \varepsilon$.

Let us prove this by induction on s. For $s = 1$ the statement is trivially true. Fix s_0 and assume that the statement holds if $s < s_0$. To prove the statement for $s = s_0$, consider the chain $\{G_j\}_{j=0}^{s}$ with $G_s = \widetilde{B}(c', 8^{\ell_{\max}} r\varepsilon)$ and $G_0 \ni x$. We distinguish two cases:

Case 1. There exists $0 \le s_1 < s$ such that G_{s_1} contains a critical point c_1. By the definition of the BC^* property, it follows that $G_{s_1} \subset \widetilde{B}(c_1, \varepsilon)$. If $s_1 = 0$, then $c_1 = c$ and $J_0 \subset G_0 \subset \widetilde{B}(c, \varepsilon)$. Otherwise, the statement follows by the induction hypothesis.

Case 2. For any $0 \leq k < s$, G_k contains no critical point. Then $f^{s-1} : G_1 \to G_s$ is a diffeomorphism. By the Macroscopic Koebe Principle (Proposition A.1 (iii)), we obtain that G_1 contains the 1-scaled neighborhood of J_1. Since $c \notin G_0$, $f(c) \notin G_1$ and $f(x) \in B_\varepsilon(f(c))$, it follows that $|J_1| < \varepsilon$. □

Proof of Theorem 8. Combine Lemma 3.6 and Proposition 3.5. □

4. Exponential mixing rates

In this section we focus on exponentially mixing acip's. For simplicity, we restrict our discussion to interval maps $f : I \to I$ that are topologically exact. This is not a serious restriction, since every non-degenerate smooth map having an acip can be decomposed into maps satisfying this property. More precisely, for every non-degenerate smooth map $f : I \to I$ having an acip ν, the support of ν can be decomposed into a finite union of cycles of intervals, in such a way that for each of these intervals I' there is an integer $n \geq 1$ such that $f^n(I') = I'$, such that the non-degenerate smooth map $f^n : I' \to I'$ is topologically exact, and such that $\nu|_{I'}$ is an acip of $f^n|_{I'}$.

Let (X, dist) be a compact metric space, $T : X \to X$ a continuous map, and ν a Borel probability measure on X that is invariant by T. Given a pair of square integrable functions φ, $\psi : X \to \mathbb{R}$, and an integer $n \geq 1$, define

$$\mathscr{C}_n(\varphi, \psi) := \int_X \varphi \circ f^n \cdot \psi \, d\nu - \int_X \varphi \, d\nu \int_X \psi \, d\nu.$$

The measure ν is (*strongly*) *mixing*, if for every pair of square integrable functions φ, $\psi : X \to \mathbb{R}$ we have

$$\mathscr{C}_n(\varphi, \psi) \to 0 \text{ as } n \to +\infty. \tag{4.1}$$

In order to quantify mixing rates, it is customary to restrict the functions φ and ψ in (4.1) to Banach spaces that are usually smaller than the space of square integrable functions. Here, we restrict our discussion to results where φ is in the space of bounded functions, and ψ in that of Lipschitz continuous functions. It is the setting of Young's general method to estimate mixing rates, see [You99] and also [BMD05, Gou04, MT12, Sar01] for extensions and refinements. For smooth interval maps, most of the results in the literature have been obtained through Young's method. There are exceptions, like for example some of the earliest results [KN92, You92] in which the function ψ in (4.1) is taken in the space of functions of bounded variation. However, in all the exceptions we are aware of, Young's method can be applied as an alternative approach.

The measure ν is *exponentially mixing*, if there are constants ρ in $(0, 1)$ and $C > 0$ such that for every bounded and measurable function $\varphi : X \to \mathbb{R}$, and every Lipschitz continuous function $\psi : X \to \mathbb{R}$, we have for every integer $n \geq 1$

$$|\mathscr{C}_n(\varphi, \psi)| \leq C\rho^n \left(\sup_X |\varphi| \right) \cdot \|\psi\|_{\mathrm{Lip}}, \tag{4.2}$$

where $\|\psi\|_{\mathrm{Lip}} = \sup_{x, x' \in X, x \neq x'} \frac{|\psi(x) - \psi(x')|}{\mathrm{dist}(x, x')}$.

In §§4.1, 4.2 we discuss exponential mixing rates for unimodal, and for multimodal maps, respectively.

4.1. **Unimodal maps.**

For interval maps, the first general result on mixing rates were obtained independently by Keller and Nowicki in [KN92], and by

Young in [You92]. To state this result, we make the following definition that will be important in what follows.

Definition 4.1. Let $f : I \to I$ be a smooth non-degenerate unimodal map having only repelling periodic points. If we denote by c the critical point of f, then f satisfies the *Collet-Eckmann Condition*, if the lower Lyapunov exponent of f at $f(c)$ is strictly positive:

$$\liminf_{n \to +\infty} \frac{1}{n} \ln |Df^n(f(c))| > 0. \tag{4.3}$$

We note that in the case f has negative Schwarzian derivative, like for example when f is a quadratic map, (4.3) implies that all periodic points of f are hyperbolic repelling.

By the results mentioned above, a smooth non-degenerate unimodal map satisfying the Collet-Eckmann condition has a unique acip. In [KN92, You92] it was shown that this acip is exponentially mixing.[§] Later, Nowicki and Sands showed in [NS98] that in fact the Collet-Eckmann condition characterizes the existence of an exponentially mixing acip.

Theorem 9 ([NS98], Theorem A)**.** *Let f be a smooth non-degenerate unimodal map that is topologically exact. Then f has an exponentially mixing acip if and only if f satisfies the Collet-Eckmann condition.*

In fact, [NS98, Theorem A] gives a long list of equivalent formulations of the Collet-Eckmann condition, of which Theorem 9 is just a sample. The proof of these equivalences in [NS98] relies on delicate combinatorial arguments that are specific to unimodal maps. A substantially simpler proof can be found in [RL12a], where these equivalences are extended to multimodal maps.

The list of equivalent formulations of the Collet-Eckmann condition in [NS98] was enhanced by Nowicki and Przytycki in [NP98], about the same time: The Collet-Eckmann condition is equivalent to a technical condition formulated in purely topological terms. A consequence of this last result is the rather surprising fact that the Collet-Eckmann condition, and therefore the existence of an exponentially mixing acip, is invariant under topological conjugacy.

The Collet-Eckmann condition is known to be abundant in various settings. In the quadratic family $(f_\lambda)_{\lambda \in (0,4]}$, there is a subset of positive measure of parameters λ for which f_λ satisfies the Collet-Eckmann condition, see [BC85, Jak81], and also [GS14, Tsu01, WY06] for similar results for more general families of interval maps. In fact, Avila and Moreira showed in [AM05], based on the results of Lyubich in [Lyu02], that there is a set of full Lebesgue measure of parameters λ in $(0, 4]$ such that either f_λ has an strictly attracting cycle, so f_λ is uniformly hyperbolic, or f_λ satisfies the Collet-Eckmann condition. See also the recent approach in [GS14], where

[§]The mixing rates obtained in these results are for test functions of bounded variation. However, these maps are exponentially mixing as defined here.

this result is obtained as a consequence of a general result for multimodal maps.

4.2. **Multimodal maps.** To discuss exponentially mixing acip's for non-degenerate smooth maps with several critical points, we first introduce the Collet-Eckmann condition.

Definition 4.2. A non-degenerate smooth map $f : I \to I$ satisfies the *Collet-Eckmann condition*, if for every critical point c of f the sequence $(|Df^n(f(c))|)_{n=1}^{+\infty}$ grows exponentially in n, and if all periodic points of f are hyperbolic repelling.

As for unimodal maps, a non-degenerate smooth interval map that is topologically exact and satisfies the Collet-Eckmann condition has an exponentially mixing acip, see [BLVS03] for the case where all the critical points have the same order, and [RLS14, Remark 2.14] for the general case. For multimodal maps, the Collet-Eckmann condition is also abundant, see [GS14, Tsu01].

In contrast with the case of unimodal maps, the Collet-Eckmann condition does not characterize the existence of an exponentially mixing acip. This follows from the examples given in [PRLS03, §6.1], together with [RLS14, Remark 2.14]. Other examples are described in Example 4.4, below. We note also that, in contrast with the unimodal case, the Collet-Eckmann condition is not invariant under topological conjugacy, see [PRLS03, Appendix C].

For multimodal maps, the existence of an exponentially mixing acip has several equivalent formulations in the spirit of [NS98], in spite of the fact that the Collet-Eckmann condition is not among them, see [RL12a]. One of the simplest equivalent formulations is the following strong form of Pesin's non-uniform hyperbolicity condition: There is $\chi > 0$ such that for every invariant Borel probability measure μ we have

$$\chi(\mu) := \int \ln |Df| \, d\mu \geq \chi.$$

We say that f is *asymptotically expanding* if it satisfies this property. From the technical viewpoint, the following equivalent formulation of asymptotic expansion is perhaps the most useful.

Definition 4.3. Let $f : I \to I$ be a non-degenerate smooth map. We say that f satisfies the *Exponential Shrinking of Components Condition* if there are constants $\delta > 0$ and $\lambda > 1$ such that for every interval J contained in I that satisfies $|J| \leq \delta$, the following property holds: For every integer $n \geq 1$ and every connected component W of $f^{-n}(J)$ we have $|W| \leq \lambda^{-n}$.

For a map satisfying the Exponential Shrinking of Components Condition, the existence of an exponentially mixing acip is shown in [PRL07] in the case of complex rational maps; the proof applies without change to a non-degenerate smooth map that is topologically exact, see [RLS14, Remark 2.14] for a proof written in this setting.

These results are summarized in the following theorem, see [RL12a] for further details.

Theorem 10. *For a non-degenerate smooth interval map f that is topologically exact, the following properties are equivalent.*

1. *f is asymptotically expanding.*
2. *f satisfies the Exponential Shrinking of Components Condition.*
3. *f as an exponentially mixing acip.*

Finally, we remark that the result of Nowicki and Przytycki in [NP98], combined with [RL12a], implies that for a non-degenerate smooth map that is topologically exact the existence of an exponentially mixing acip is invariant under topological conjugacy, see [RL12a, §1.4].

Example 4.4. The following are examples of non-degenerate smooth maps having an exponentially mixing acip, but that do not satisfy the Collet-Eckmann condition.

For γ in $(0,1]$, consider the real polynomial

$$g_\gamma(x) := \frac{27}{4\gamma^3} x(x-\gamma)^2.$$

Note that the critical points of g_γ are $c(\gamma) := \gamma/3$ and $\widetilde{c}(\gamma) := \gamma$. Furthermore, g_γ maps $\widetilde{c}(\gamma)$ to $x = 0$, which is a hyperbolic repelling fixed point of g_γ, and that g_γ maps $c(\gamma)$ to $x = 1$. Finally, note that $g_\gamma(1) > 0$ and that when $g_\gamma(1) \le 1$, the map g_γ maps $[0,1]$ to itself.

For a first example, let γ_0 in $(0,1)$ be such that $g_{\gamma_0}(1) = \widetilde{c}(\gamma_0)$. Then g_{γ_0} does not satisfy the Collet-Eckmann condition because in this case $g^2_{\gamma_0}(c(\gamma_0)) = \widetilde{c}(\gamma_0)$, and therefore for every $n \ge 1$ we have $Dg^n_{\gamma_0}(g(\gamma_0)(c(\gamma_0))) = 0$. It is easy to see that g_{γ_0} is topologically exact on $[0,1]$. Moreover, g_{γ_0} has an exponentially mixing acip by [RLS14, Remark 2.14]. The existence of an exponentially mixing acip can be shown more directly in this case, using the Markov partition of g_{γ_0} formed by the intervals

$$[0, c(\gamma_0)], [c(\gamma_0), \widetilde{c}(\gamma_0)], \text{ and } [\widetilde{c}(\gamma_0), 1],$$

and using the fact that g_{γ_0} is uniformly expanding with respect to a singular metric on $[0,1]$.

A more subtle example can be constructed by choosing the parameter γ so that the forward orbit of $c(\gamma)$ under g_γ approaches $\widetilde{c}(\gamma)$ very closely at certain special times, without hitting $\widetilde{c}(\gamma)$ exactly. If at this special times the approximation is super-exponential with respect to time, then the Collet-Eckmann condition fails for g_γ. If in addition the forward orbit of $c(\gamma)$ under stays at a positive distance from $c(\gamma)$, then g_γ has an acip by [Mis81]. It follows from [RLS14, Remark 2.14] that this acip is exponentially mixing.

Appendix A. The Koebe principle

Proposition A.1 (Koebe principle)**.** *For any $f \in \mathcal{A}$, there exists $\eta(f) > 0$ such that the following holds. Let $s \geq 1$ be an integer and let $T = (a, b)$ be an interval. Assume that $f^s|T$ is a diffeomorphism onto its image and that $|f^s(T)| < \eta(f)$. Then*

(i) *(the Minimum Principle) for every $x \in T$,*
$$|Df^s(x)| \geq 0.9 \min\left(|Df^s(a)|, |Df^s(b)|\right);$$
(ii) *(the one-sided Koebe Principle) Let $x \in T$ be such that $|f^s(a) - f^s(x)| \geq \tau |f^s(x) - f^s(b)|$. Then*
$$|Df^s(x)| \geq 0.9 \left(\frac{\tau}{1+\tau}\right)^2 |Df^s(b)|,$$
(iii) *(the Koebe Principle) If J is a subinterval of T such that $f^s(J)$ is τ-well inside $f^s(T)$, then for any $x, y \in J$,*
$$0.9 \left(\frac{\tau}{1+\tau}\right)^2 \leq \frac{|Df^s(x)|}{|Df^s(y)|} \leq \frac{1}{0.9}\left(\frac{1+\tau}{\tau}\right)^2;$$
(iv) *(the Macroscopic Koebe Principle) If J is a subinterval of T such that $f^s(J)$ is τ-well inside $f^s(T)$, then J is τ'-well inside T, where $\tau' = 0.9\tau^2/(1+2\tau)$.*

Proof. Let us say that a diffeomorphism φ between intervals is almost linear if for any x, y in its domain, we have $|D\varphi(x)| \geq 0.9|D\varphi(y)|$. It suffices to prove that there exists $0 \leq s_0 < s$ such that

- $f^{s-s_0} : f^{s_0}(T) \to f^s(T)$ is almost linear;
- either $s_0 = 0$ or $f^{s_0} : T \to f^{s_0}(T)$ has negative Schwarzian.

By the third statement of Theorem C in [vSV04], there exists a neighborhood U of $\mathrm{Crit}(f)$ such that for any $x \in X$ and $n \geq 0$ with $f^n(x) \in U$, we have $Sf^{n+1}(x) < 0$, where $S\phi$ denotes the Schwarzian derivative of ϕ. Let $V \Subset U$ be a smaller neighborhood of $\mathrm{Crit}(f)$. Provided that $|f^s(T)|$ is small enough, $\max_{i=0}^{s} |f^i(T)| < d(\partial U, \partial V)$, so that for each $i \in \{0, 1, \ldots, s\}$, either $f^i(T) \subset U$ or $f^i(T) \cap V = \emptyset$. Let $s_0 \in [0, s)$ be minimal such that $f^i(T) \cap V = \emptyset$ for all $s_0 \leq i < s$. Then either $s_0 = 0$ or $f^{s_0 - 1}(T) \subset U$ so that $f^{s_0} : T \to f^{s_0}(T)$ has negative Schwarzian derivative. Moreover, by Mañé's theorem, f is uniformly expanding outside V. Thus provided that $|f^s(T)|$ is small enough, $f^{s-s_0} : f^{s_0}(T) \to f^s(T)$ is almost linear. □

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